The Mathematical Association of America
American Mathematics Competitions

23rd Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

Tuesday, March 8, 2005

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given in your school on TUESDAY and WEDNESDAY, April 19 & 20, 2005.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

The publication, reproduction or communication of the problems or solutions of the AIME during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, email, world wide web, or media of any type is a violation of the competition rules.

1. Six congruent circles form a ring with each circle externally tangent to the two circles adjacent to it. All six circles are internally tangent to a circle $\mathcal{C}$ with radius 30. Let K be the area of the region inside $\mathcal{C}$ and outside all of the six circles in the ring. Find $\lfloor K \rfloor$. (The notation $\lfloor K \rfloor$ denotes the greatest integer that is less than or equal to K.)

2. For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, \ldots$. For how many values of k does S_k contain the term 2005?

3. How many positive integers have exactly three proper divisors, each of which is less than 50? (A *proper divisor* of a positive integer n is a positive integer divisor of n other than n itself.)

4. The director of a marching band wishes to place the members into a formation that includes all of them and has no unfilled positions. If they are arranged in a square formation, there are 5 members left over. The director finds that if they are arranged in a rectangular formation with 7 more rows than columns, the desired result can be obtained. Find the maximum number of members this band can have.

5. Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of a face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

6. Let P be the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$. (The notation $\lfloor P \rfloor$ denotes the greatest integer that is less than or equal to P.)

7. In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $m\angle A = m\angle B = 60^\circ$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.

8. The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Given that their sum is m/n, where m and n are relatively prime positive integers, find $m + n$.

9. Twenty-seven unit cubes are each painted orange on a set of four faces so that the two unpainted faces share an edge. The 27 cubes are then randomly arranged to form a $3 \times 3 \times 3$ cube. Given that the probability that the entire surface of the larger cube is

orange is $\dfrac{p^a}{q^b r^c}$, where p, q, and r are distinct primes and a, b, and c are positive integers,

find $a + b + c + p + q + r$.

10. Triangle ABC lies in the Cartesian plane and has area 70. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively, and the coordinates of A are (p, q). The line containing the median to side $\overline{BC}$ has slope -5. Find the largest possible value of $p + q$.

11. A semicircle with diameter d is contained in a square whose sides have length 8. Given that the maximum value of d is $m - \sqrt{n}$, where m and n are integers, find $m + n$.

12. For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by

$$S(n) = \tau(1) + \tau(2) + \cdots + \tau(n).$$

Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

13. A particle moves in the Cartesian plane from one lattice point to another according to the following rules:

• From any lattice point $(a,\ b)$, the particle may move only to $(a+1,\ b)$, $(a,\ b+1)$, or $(a+1,\ b+1)$.

• There are no right angle turns in the particle's path. That is, the sequence of points visited contains neither a subsequence of the form $(a,\ b)$, $(a+1,\ b)$, $(a+1,\ b+1)$ nor a subsequence of the form $(a,\ b)$, $(a,\ b+1)$, $(a+1,\ b+1)$.

How many different paths can the particle take from $(0,\ 0)$ to $(5,\ 5)$?

14. Consider the points $A(0, 12)$, $B(10, 9)$, $C(8, 0)$, and $D(-4, 7)$. There is a unique square $\mathcal{S}$ such that each of the four points is on a different side of $\mathcal{S}$. Let K be the area of $\mathcal{S}$. Find the remainder when $10K$ is divided by 1000.

15. In $\triangle ABC$, $AB = 20$. The incircle of the triangle divides the median containing C into three segments of equal length. Given that the area of $\triangle ABC$ is $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime, find $m + n$.

Your Exam Manager will receive a copy of the 2005 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:*

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2005 USAMO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held in your school on Tuesday, April 19 & Wednesday, April 20. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC - - http://www.unl.edu/amc/d-publication/publication.html
MAA -- https://enterprise.maa.org/ecomtpro/timssnet/common/tnt_frontpage.cfm