THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

24th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME I)

Tuesday, March 7, 2006

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**

4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given in your school on TUESDAY and WEDNESDAY, April 18 & 19, 2006.

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

The publication, reproduction or communication of the problems or solutions of the AIME during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, email, world wide web, or media of any type is a violation of the competition rules.

1. In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

2. Let set $\mathcal{A}$ be a 90-element subset of $\{1, 2, 3, \ldots, 100\}$, and let S be the sum of the elements of $\mathcal{A}$. Find the number of possible values of S.

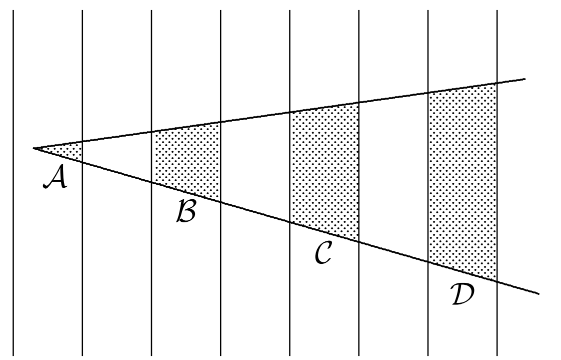
3. Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is 1/29 of the original integer.

4. Let N be the number of consecutive 0's at the right end of the decimal representation of the product $1!\,2!\,3!\,4!\cdots 99!\,100!$. Find the remainder when N is divided by 1000.

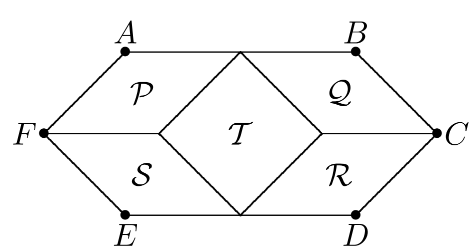
5. The number
$$\sqrt{104\sqrt{6} + 468\sqrt{10} + 144\sqrt{15} + 2006}$$
can be written as $a\sqrt{2} + b\sqrt{3} + c\sqrt{5}$, where a, b, and c are positive integers. Find $a \cdot b \cdot c$.

6. Let $\mathcal{S}$ be the set of real numbers that can be represented as repeating decimals of the form $0.\overline{abc}$ where a, b, c are distinct digits. Find the sum of the elements of $\mathcal{S}$.

7. An angle is drawn on a set of equally spaced parallel lines as shown. The ratio of the area of shaded region $\mathcal{C}$ to the area of shaded region $\mathcal{B}$ is 11/5. Find the ratio of the area of shaded region $\mathcal{D}$ to the area of shaded region $\mathcal{A}$.

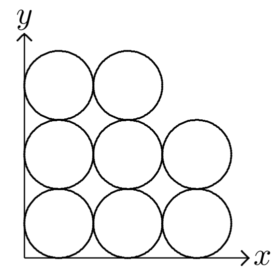


8. Hexagon $ABCDEF$ is divided into five rhombuses, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$, as shown. Rhombuses $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ are congruent, and each has area $\sqrt{2006}$. Let K be the area of rhombus $\mathcal{T}$. Given that K is a positive integer, find the number of possible values for K.



9. The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

10. Eight circles of diameter 1 are packed in the first quadrant of the coordinate plane as shown. Let region $\mathcal{R}$ be the union of the eight circular regions. Line ℓ, with slope 3, divides $\mathcal{R}$ into two regions of equal area. Line ℓ's equation can be expressed in the form $ax = by + c$, where a, b, and c are positive integers whose greatest common divisor is 1. Find $a^2 + b^2 + c^2$.



11. A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \le k \le 8$. A tower is to be built using all 8 cubes according to the rules:

- Any cube may be the bottom cube in the tower.
- The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Let T be the number of different towers that can be constructed. What is the remainder when T is divided by 1000?

12. Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8 \cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

13. For each even positive integer x, let $g(x)$ denote the greatest power of 2 that divides x. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer n, let $S_n = \sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer n less than 1000 such that S_n is a perfect square.

14. A tripod has three legs each of length 5 feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is 4 feet from the ground. In setting up the tripod, the lower 1 foot of one leg breaks off. Let h be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then h can be written in the form $\frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $\lfloor m + \sqrt{n} \rfloor$. (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

15. Given that a sequence satisfies $x_0 = 0$ and $|x_k| = |x_{k-1} + 3|$ for all integers $k \ge 1$, find the minimum possible value of $|x_1 + x_2 + \cdots + x_{2006}|$.

Your Exam Manager will receive a copy of the 2006 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2006 USAMO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held in your school on Tuesday, April 18 & Wednesday, April 19. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC - - http://www.unl.edu/amc/d-publication/publication.html
MAA -- https://enterprise.maa.org/ecomtpro/timssnet/common/tnt_frontpage.cfm

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