The Mathematical Association of America
American Mathematics Competitions

26th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 18, 2008

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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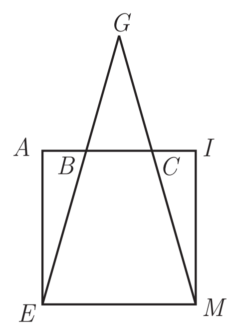
1. (Answer: 252)

   Suppose at the beginning there are x students at the party. Then $0.6x$ students are girls, and $0.4x$ students like to dance. When the 20 boys join the party, there are $0.58(x+20)$ girls at the party, which must equal the original number of girls at the party. Thus $0.58(x+20) = 0.6x$, which implies $0.58 \cdot 20 = (0.6 - 0.58)x = 0.02x$, and $x = 580$. Thus the current number of students who like to dance is $0.40 \cdot 580 + 20 = 252$.

2. (Answer: 025)



   Let h be the required altitude, and let B and C be the points of intersection of $\overline{AI}$ with $\overline{GE}$ and $\overline{GM}$, respectively. Then the fact that $\triangle GEM$ is similar to $\triangle GBC$ implies that $\frac{h-10}{h} = \frac{BC}{10}$. Thus $BC = \frac{10h-100}{h}$, and the area common to GEM and $AIME$ equals $\frac{1}{2} \cdot 10h - \frac{1}{2}(\frac{10h-100}{h})(h-10) = 80$. This equation reduces to $5h^2 - 5(h-10)^2 = 80h$, or $100h - 500 = 80h$. The required length is then 25.

3. (Answer: 314)

   Let $b =$ Ed and Sue's biking rate, let $j =$ their jogging rate, and let $s =$ their swimming rate. Then

$$2b + 3j + 4s = 74$$
$$4b + 2j + 3s = 91.$$

   Adding -2 times the first equation to the second equation and solving for j yields $j = \frac{57}{4} - \frac{5}{4}s$. The only ordered pairs (s, j) that satisfy this equation in which both s and j are positive integers are $(1, 13)$, $(5, 8)$, and $(9, 3)$. However, subtracting the first equation from the second equation and solving for b yields $b = \frac{s+j+17}{2}$, and only the ordered pair $(5, 8)$ produces an integer value (of 15) for b. Thus the sum of the squares of Ed's rates is $15^2 + 8^2 + 5^2 = 314$.

4. (Answer: 080)

Observe that if x is a positive integer, then

$$(x+42)^2 = x^2+84x+1764 < x^2+84x+2008 < x^2+90x+2025 = (x+45)^2.$$

Because $x^2 + 84x + 2008$ is a square, it is either $(x + 43)^2$ or $(x + 44)^2$. In the first case, $2x = 159$, which is impossible, and in the second case, $4x = 72$, which implies $x = 18$. In that case, $y = x + 44 = 62$, and $x + y = 18 + 62 = 80$.

OR

Complete the square in x to find that $x^2 + 84x + 1764 = (x + 42)^2 = y^2 - 244$. Letting $v = x + 42$, the condition $y^2 - v^2 = 244$ is equivalent to $(y - v)(y + v) = 2 \cdot 2 \cdot 61$, whose only positive integer solutions are given by $y - v = 2$ and $y + v = 2 \cdot 61$. Thus the only two perfect squares that differ by 244 are 60^2 and 62^2. Hence $x = 60 - 42 = 18$, and $y = 62$.

5. (Answer: 014)

The slant height of the cone is $\sqrt{r^2 + h^2}$. This is the radius of the circle described by the cone as it rolls on the table. The circumference of this circle is $2\pi\sqrt{r^2 + h^2}$. The circumference of the base of the cone is $2\pi r$. The cone makes 17 complete rotations in rolling back to its original position, and so

$$17 = \frac{2\pi\sqrt{r^2 + h^2}}{2\pi r} = \sqrt{1 + \left(\frac{h}{r}\right)^2}.$$

Thus

$$\frac{h}{r} = \sqrt{17^2 - 1} = 12\sqrt{2},$$

and $m + n = 14$.

6. (Answer: 017)

Note that row r is an arithmetic progression with common difference 2^r. It can be shown by induction that the first element of row r is $r(2^{r-1})$, and thus the nth element of row r is $r(2^{r-1})+(n-1)2^r = 2^{r-1}(r+2n-2)$, for $1 \le n \le 51 - r$. For the element $2^{r-1}(r + 2n - 2)$ to be a multiple of 67, it must be true that $r + 2n - 2$ is a multiple of 67, or that $2n \equiv 2 - r \pmod{67}$. Multiplying both sides of this congruence by 34 yields $68n \equiv 68 - 34r \pmod{67}$, and the fact that $68 \equiv 1 \pmod{67}$ and $-34 \equiv 33 \pmod{67}$ produces $n \equiv 1 + 33r \pmod{67}$. Now in odd-numbered rows, the equations $r = 2k - 1$ and $n \equiv 1 + 33r \equiv 66k - 32 \equiv 35 - k \pmod{67}$ yield $(k, r, n) = (1, 1, 34), (2, 3, 33), (3, 5, 32), \ldots, (17, 33, 18)$. Note that $k > 17$ does not produce any solutions, because row $2k - 1$ has fewer than $35 - k$ entries for $k > 17$. In even-numbered rows, the equations

$r = 2k$ and $n \equiv 1 + 33r \equiv 66k + 1 \equiv 68 - k \pmod{67}$ yield $(k, r, n) = (1, 2, 67), (2, 4, 66), \ldots, (25, 50, 43)$, none of which correspond to entries in the array, so there are no solutions in the even-numbered rows. Thus there are 17 multiples of 67 in the array.

OR

Create a reduced array by dividing each new row by the largest common factor of the elements in the row. The first few rows of this reduced array will be

$$\begin{array}{ccccccccccccccccc}
1 & & 3 & & 5 & & 7 & & \ldots & & 93 & & 95 & & 97 & & 99 \\
& 1 & & 2 & & 3 & & 4 & \ldots & 46 & & 47 & & 48 & & 49 & \\
& & 3 & & 5 & & 7 & & \ldots & & 93 & & 95 & & 97 & & \\
& & & 2 & & 3 & & 4 & \ldots & 46 & & 47 & & 48 & & & \\
& & & & 5 & & 7 & & \ldots & & 93 & & 95. & & & &
\end{array}$$

Note that in row n, if $n = 2k + 1$, then the elements of row $n + 1$ will have the form $[(2i + 1) + (2i + 3)]/4 = i + 1$. If $n = 2k$, then the elements of row $n + 1$ will have the form $[i + (i + 1)] = 2i + 1$. Thus each odd-numbered row consists of the consecutive odd integers from n to $100 - n$, and each even-numbered row consists of the consecutive integers from $n/2$ to $50 - n/2$. Furthermore, because 67 is not divisible by 4, the entries in the reduced array which are multiples of 67 correspond exactly to the entries in the original array which are multiples of 67. The additive definition of the array combined with the distributive property of multiplication ensures this. Because all elements of the reduced array are between 1 and 99, the only elements of the array divisible by 67 are those equal to 67. All elements of the even-numbered row n are at most $50 - n/2$, so even-numbered rows contain no multiples of 67, and elements of the odd-numbered row n are at most $100 - n$, so the odd-numbered rows up to 33 contain exactly one 67. Thus the reduced array contains 17 entries equal to 67, and the original array contains 17 multiples of 67.

7. (Answer: 708)

Note that $1^2 = 1 \in S_0$. All sets S_i contain a perfect square unless i is so large that, for some integer a, $a^2 \in S_j$ with $j < i$, and $(a + 1)^2 \in S_k$ with $k > i$. This implies that $(a + 1)^2 - a^2 > 100$, and hence $a \geq 50$. But $50^2 = 2500$, which is a member of S_{25}, so all sets $S_0, S_1, S_2, \ldots, S_{25}$ contain at least one perfect square. Furthermore, for all $i > 25$, each set S_i which contains a perfect square can only contain one such square. The largest value in any of the given set is 99999, and $316^2 < 99999 < 317^2$. Disregarding the first 50 squares dealt with above, there are $316 - 50 = 266$ perfect squares, each the sole perfect square member of one of the 974 sets

$S_{26}, S_{27}, S_{28}, \ldots, S_{999}$. Thus there are $974 - 266$ sets without a perfect square, and the answer is 708.

8. (Answer: 047)

   Applying the addition formula for tangent to $\tan(\arctan x + \arctan y)$ results in the formula $\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$, which is valid for $0 < xy < 1$. Thus $\arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5} = \arctan \frac{7}{11} + \arctan \frac{1}{5} = \arctan \frac{23}{24}$. The left-hand side of the original equation is therefore equivalent to $\arctan \frac{23n+24}{24n-23}$. Because this must equal $\arctan 1$, it follows that $n = 47$.

9. (Answer: 190)

   Because each crate has three different possible heights, there are 3^{10} equally likely ways to stack the ten crates. Suppose a stack of crates is 41 ft tall. Let x, y, and z be the number of crates with heights 3, 4, and 6 ft, respectively. Then $x+y+z = 10$ and $3x+4y+6z = 41$. This system is equivalent to the system $x - 2z = -1$ and $y + 3z = 11$, or $x = 2z - 1$, $y = 11 - 3z$. Because $x \geq 0$ and $y \geq 0$, $z \in \{1, 2, 3\}$, with each value of z yielding a different solution (x, y, z):

   $$(1, 8, 1), (3, 5, 2), (5, 2, 3).$$

   For each ordered triple (x, y, z) there are $\frac{10!}{x!y!z!}$ equally likely ways to stack the crates. Thus the number of ways to stack the crates is $\frac{10!}{1!8!1!} + \frac{10!}{3!5!2!} + \frac{10!}{5!2!3!} = 5130$. The desired probability is $\frac{5130}{3^{10}} = \frac{190}{3^7}$, so the required value of m is 190.

   Note: Another way to find the ordered triples which solve the two given equations is to notice that a stack of 10 crates must be at least 30 ft tall. Each y crate adds 1 additional foot to the height, and each z crate adds 3 additional feet. The stack requires 11 more feet than the minimum of 30 feet, so $y + 3z = 11$.

10. (Answer: 032)

    Because $30\sqrt{7} = DE \leq DA + AE = DA + 10\sqrt{7}$, it follows that $DA \geq 20\sqrt{7}$. Let $\theta = \angle DCA$. Applying the Law of Sines to $\triangle DCA$ yields $\frac{10\sqrt{21}}{\sin(\pi/3)} = \frac{DA}{\sin\theta} \geq \frac{20\sqrt{7}}{\sin\theta}$, which implies $\sin\theta \geq 1$. Then θ must be $\frac{\pi}{2}$, $DA = 20\sqrt{7}$, and point E lies on the extension of side $\overline{DA}$. Applying the Pythagorean Theorem to $\triangle DCA$ yields $DC = \sqrt{DA^2 - CA^2} = 10\sqrt{7}$. Then $DF = DC \cdot \cos\frac{\pi}{3} = 5\sqrt{7}$. Therefore $EF = DE - DF = 30\sqrt{7} - 5\sqrt{7} = 25\sqrt{7}$, and $m + n = 32$.

11. (Answer: 172)

    Let a_n and b_n be the number of permissible sequences of length n beginning with A and B, respectively, and let $x_n = a_n + b_n$ be the total number of permissible sequences of length n. Any permissible sequence of length $n+2$ that begins with A must begin with AA followed by a permissible sequence of length n, so that $a_{n+2} = x_n$ for $n \geq 1$. Any permissible sequence of length $n+2$ that begins with B must begin with a single B followed by a permissible sequence of length $n+1$ beginning with A, or else it must begin with BB followed by a permissible sequence of length n beginning with B. Thus $b_{n+2} = a_{n+1} + b_n$ for $n \geq 1$, and hence $b_{n+2} = a_{n+1} + x_n - a_n = x_n + x_{n-1} - x_{n-2}$. Because $b_{n+2} = a_{n+1} + b_n$, it follows that

    $$x_n + x_{n-1} - x_{n-2} = x_{n-1} + (x_{n-2} + x_{n-3} - x_{n-4}),$$

    and so $x_n = 2x_{n-2} + x_{n-3} - x_{n-4}$ for $n \geq 5$. Note that the permissible sequences of length at most 4 are B, AA, AAB, BAA, BBB, $AAAA$, and $BAAB$. Thus $x_1 = x_2 = 1$, $x_3 = 3$, and $x_4 = 2$. Applying these results to the recursion given above produces the sequence 1, 1, 3, 2, 6, 6, 11, 16, 22, 37, 49, 80, 113, and 172, and the required value is 172.

12. (Answer: 375)

    Let s be the speed of the cars in kilometers per hour. Then the number of car lengths between consecutive cars is $\lceil s/15 \rceil$(the least integer $\geq \frac{s}{15}$), so the distance between consecutive cars is $4\lceil s/15 \rceil$ meters. Thus the distance from the front of one car to the front of the next is $d = 4\lceil s/15 \rceil + 4$ meters. In one hour each car travels $1000s$ meters. Let the interval from the front of one car to the front of the next be called a *gap*. Then the number N of gaps that pass the eye in one hour is

    $$\frac{1000s}{d} = \frac{1000s}{4\lceil s/15 \rceil + 4} = \frac{250s}{\lceil s/15 \rceil + 1}.$$

    Let $\lceil s/15 \rceil = \frac{s}{15} + \varepsilon$, where $0 \leq \varepsilon < 1$. Then

    $$N = \frac{250s}{\frac{s}{15} + \varepsilon + 1} = \frac{3750}{1 + \frac{15\varepsilon + 15}{s}}.$$

    The quantity $\dfrac{15\varepsilon + 15}{s}$ is positive and approaches zero as s increases. Thus $N \leq 3750$, and by taking s sufficiently large, N can be made as close to 3750 as desired. Therefore, at a high enough speed, 3749.9 gaps will pass the eye in one hour. Assuming that at the start of the hour the eye is exactly even with the front of a car, it will be passed by one car in each of the ensuing 3749 gaps, and by one additional car at the beginning of the last 0.9 of a gap. Thus the eye will be passed by 3750 cars. The quotient when 3750 is divided by 10 is 375.

Note: With a little more calculation, it can be shown that the minimum speed for which 3750 cars pass the eye is $s = 56250$ kph. Do not try this at home!

13. (Answer: 040)

Applying the first five conditions in order yields $a_0 = 0$, and then $a_1 + a_3 + a_6 = 0, a_1 - a_3 + a_6 = 0, a_2 + a_5 + a_9 = 0, a_2 - a_5 + a_9 = 0$, which imply that $a_3 = a_5 = 0$ and $a_6 = -a_1, a_9 = -a_2$. Thus $p(x, y) = a_1(x - x^3) + a_2(y - y^3) + a_4xy + a_7x^2y + a_8xy^2$. Similarly, the next two conditions imply that $a_8 = 0$ and $a_7 = -a_4$, so that $p(x, y) = a_1(x - x^3) + a_2(y - y^3) + a_4(xy - x^2y)$. The last condition implies that $-6a_1 - 6a_2 - 4a_4 = 0$, so that $a_4 = -\frac{3}{2}(a_1 + a_2)$. Thus $p(x, y) = a_1(x - x^3 - \frac{3}{2}(xy - x^2y)) + a_2(y - y^3 - \frac{3}{2}(xy - x^2y))$. If $p(r, s) = 0$ for every such polynomial, then

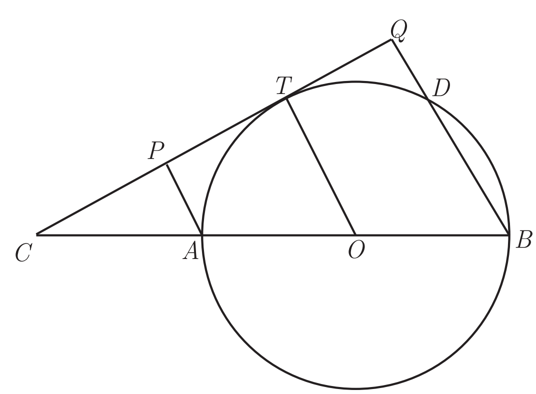
$$0 = r - r^3 - \frac{3}{2}(rs - r^2s) = \frac{1}{2}r(r - 1)(3s - 2r - 2), \text{ and}$$
$$0 = s - s^3 - \frac{3}{2}(rs - r^2s) = \frac{1}{2}s(2 - 2s^2 - 3r + 3r^2).$$

The solutions to the first equation are $r = 0$ or 1, or $s = \frac{2}{3}(r + 1)$. Substituting $r = 0$ or 1 into the second equation implies that $s = 0$, 1, or -1. If $s = 0$ and $3s - 2r - 2 = 0$, then $r = -1$. These solutions represent the first seven points. Finally, if $s = 0$ and $s = \frac{2}{3}(r + 1)$, the second equation reduces to

$$0 = 2 - 2s^2 - 3r + 3r^2 = 2 - 2 \cdot \frac{4}{9}(r^2 + 2r + 1) - 3r + 3r^2 = \frac{10 - 43r + 19r^2}{9}.$$

Because $10 - 43r + 19r^2 = (2 - r)(5 - 19r)$, it follows that $r = 2$ or $r = \frac{5}{19}$. In the first case $s = 2$, which produces the point $(2, 2)$. The second case yields $s = \frac{16}{19}$. Thus $a + b + c = 5 + 16 + 19 = 40$.

14. (Answer: 432)



Let O be the center of the circle, and let Q be the foot of the perpendicular from B to line CT. Segment BQ meets the circle again (other than at B) at D. Since $\overline{AB}$ is a diameter, $\angle ADB = 90^\circ$ and $ADQP$ is a rectangle. Then

$$\begin{aligned} BP^2 &= PQ^2 + BQ^2 = AD^2 + BQ^2 = AB^2 - BD^2 + BQ^2 \\ &= AB^2 + (BQ - BD)(BQ + BD) = AB^2 + DQ(BQ + BD). \end{aligned}$$

Note that $ABQP$ is a trapezoid with $\overline{OT}$ as its midline. Hence $AB = 2OT = AP + BQ = DQ + BQ$. Consequently

$$\begin{aligned} BQ + BD &= BQ + BQ - DQ = 2BQ - DQ = 2(BQ + DQ) - 3DQ \\ &= 2AB - 3DQ. \end{aligned}$$

Combining the above shows that

$$BP^2 = AB^2 + DQ(2AB - 3DQ) = AB^2 + \frac{4 \cdot (3DQ)(2AB - 3DQ)}{12}.$$

For real numbers x and y, $4xy \le (x + y)^2$. Hence

$$BP^2 \le AB^2 + \frac{(3DQ + 2AB - 3DQ)^2}{12} = \frac{4AB^2}{3} = \frac{1296}{3}, \text{ which equals } 432.$$

This maximum may be obtained by setting $3DQ = 2AB - 3DQ$, or $3PA = 3DQ = AB = 2OT$. It follows that $\frac{AC}{CO} = \frac{PA}{OT} = \frac{2}{3}$, implying that $CA = AB$.

OR

Let $\angle ABQ = \alpha$. Let R be on $\overline{BQ}$ such that $\overline{OR} \perp \overline{BQ}$. Let r be the radius of the circle. To maximize BP^2, observe that $BP^2 = PQ^2 + BQ^2$, and from $\triangle BRO$ it follows that $OR = r \sin \alpha$ and $BR = r \cos \alpha$. Thus $PQ = 2r \sin \alpha$, and $BQ = r + r \cos \alpha$. Then

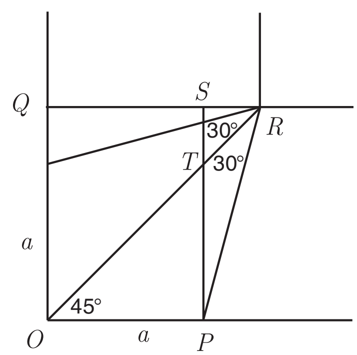
$$\begin{aligned} PB^2 &= (2r \sin \alpha)^2 + (r + r \cos \alpha)^2 \\ &= r^2 + r^2(4 \sin^2 \alpha + 1 + 2 \cos \alpha + \cos^2 \alpha) \\ &= r^2(5 - 3 \cos^2 \alpha + 2 \cos \alpha) \\ &= -3r^2 \left(-\frac{5}{3} + \cos^2 \alpha - \frac{2}{3} \cos \alpha \right) \\ &= -3r^2 \left(\left(\cos \alpha - \frac{1}{3} \right)^2 - \frac{16}{9} \right). \end{aligned}$$

The maximum for PB^2 is thus attained when $\cos \alpha = \frac{1}{3}$ and has the value $9^2(16/3) = 432$.

Query: Segment BP reaches its maximum length when lines AD, BP, and OT are concurrent. Why?

15. (Answer: 871)

Place the paper in the Cartesian plane with a corner at the origin and two of the sides on the positive coordinate axes. Label the origin as O. Let P be the point on the x-axis from which the cut starts, R be the point of intersection of the two cuts made near this corner, Q the foot of the perpendicular from R to the y-axis, S the foot of the perpendicular from P to $\overline{QR}$, and T the point of intersection of $\overline{PS}$ and $\overline{OR}$ (see the accompanying figure).



Let $OP = a = \sqrt{17}$. When the left and bottom edges of the paper are folded up to create two sides of the tray, the cut edges will meet, by symmetry, above the line $y = x$, that is, above segment OR. As the bottom edge folds up, point P traces a circular arc, with the arc centered at S and of radius SP. When the cut edges of the two sides meet, P will be at the point P' directly above T. Because $\overline{P'T}$ is perpendicular to the xy-plane, $P'T$ is the height of the tray.
Observe that $\angle ROP = 45°$. Applying the Law of Sines in $\triangle OPR$ yields

$$PR = \sin 45° \frac{OP}{\sin 30°} = a\sqrt{2}.$$

Then in right triangle PRS,

$$\begin{aligned} SP &= PR \sin 75° = PR \sin(45° + 30°) \\ &= PR(\sin 45° \cos 30° + \sin 30° \cos 45°) \\ &= a\left(\frac{\sqrt{3}+1}{2}\right). \end{aligned}$$

It then follows that

$$ST = SP - PT = a\left(\frac{\sqrt{3}+1}{2}\right) - a = a\left(\frac{\sqrt{3}-1}{2}\right).$$

Next apply the Pythagorean Theorem to right triangle $P'ST$ to find that

$$P'T^2 = P'S^2 - ST^2 = PS^2 - ST^2 = a^2\left(\frac{\sqrt{3}+1}{2}\right)^2 - a^2\left(\frac{\sqrt{3}-1}{2}\right)^2 = a^2\sqrt{3},$$

so $P'T = a\sqrt[4]{3}$. Hence the height of the tray is $\sqrt{17}\sqrt[4]{3} = \sqrt[4]{867}$, and so $m + n = 4 + 867 = 871$.