Mathematical Association of America
American Mathematics Competitions

28th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME I)

SOLUTIONS PAMPHLET

Tuesday, March 16, 2010

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 107)

   The prime factorization of 2010^2 is $2^2 \cdot 3^2 \cdot 5^2 \cdot 67^2$. Thus there are $(2+1)^4 = 81$ positive divisors of 2010^2. A perfect square divisor of 2010^2 has the form $2^w 3^x 5^y 67^z$, where each of w, x, y, z is either 0 or 2. Thus there are a total of $2^4 = 16$ perfect square divisors of 2010^2. The requested probability is therefore
   $$p = \frac{16 \cdot (81 - 16)}{\binom{81}{2}} = \frac{16 \cdot 65}{81 \cdot 40} = \frac{26}{81},$$
   and $m + n = 26 + 81 = 107$.

2. (Answer: 109)

   Let N be the product in the problem. Then
   $$N \equiv 9 \cdot 99(-1)^{997} \equiv -891 \equiv 109 \pmod{1000}.$$
   Thus the desired remainder is 109.

3. (Answer: 529)

   The conditions imply that
   $$\left(\frac{3}{4}x\right)^x = x^{\frac{3}{4}x},$$
   and hence $\pm\frac{3}{4}x = x^{\frac{3}{4}}$, or $\pm\frac{3}{4} = x^{-\frac{1}{4}}$. Thus $x = \left(\pm\frac{4}{3}\right)^4 = \frac{256}{81}$, and $y = \frac{64}{27}$. Then
   $$x + y = \frac{256}{81} + \frac{64}{27} = \frac{256 + 192}{81} = \frac{448}{81},$$
   and the requested sum is 529.

   Note: This problem goes back to Euler. For the history, see the paper by Bennett and Reznick: "Positive rational solutions to $x^y = y^{mx}$: a number-theoretic excursion", *Amer. Math. Monthly*, 111 (2004), 13–21. The positive rational solutions to $x^y = y^x$ are precisely
   $$\{(x_n, y_n)\} = \left\{\left(\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^{n+1}\right)\right\}$$
   for positive integers n.

4. (Answer: 515)

Let $p(h)$ be the probability that Jackie flips h heads. Then

$$p(0) = \left(\frac{1}{2}\right)^2 \cdot \frac{3}{7} = \frac{3}{28},$$
$$p(1) = 2 \cdot \frac{1}{4} \cdot \frac{3}{7} + \frac{1}{4} \cdot \frac{4}{7} = \frac{5}{14},$$
$$p(2) = \frac{1}{4} \cdot \frac{3}{7} + 2 \cdot \frac{1}{4} \cdot \frac{4}{7} = \frac{11}{28}, \text{ and}$$
$$p(3) = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}.$$

The probability that Jackie and Phil flip exactly the same number of heads is $[p(0)]^2 + [p(1)]^2 + [p(2)]^2 + [p(3)]^2 = \left(\frac{3}{28}\right)^2 + \left(\frac{5}{14}\right)^2 + \left(\frac{11}{28}\right)^2 + \left(\frac{1}{7}\right)^2 = \frac{123}{392}$, and the requested sum is $123 + 392 = 515$.

5. (Answer: 501)

Note that

$$a^2 - b^2 + c^2 - d^2 = (a-b)(a+b) + (c-d)(c+d) = a + b + c + d,$$

and thus $a-b = c-d = 1$. Hence $2010 = a+(a-1)+c+(c-1)$, so $a+c = 1006$. The condition $a > c$ implies that $a \geq 504$, and the condition $c > d$ implies that $c \geq 2$, so that $a \leq 1004$. For each integer k with $0 \leq k \leq 500$, the ordered quadruple $(a, b, c, d) = (504 + k, 503 + k, 502 - k, 501 - k)$ satisfies the required conditions, and thus the number of possible values of a is 501.

6. (Answer: 406)

Completing the square yields

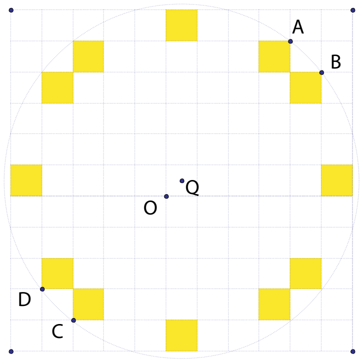
$$(x-1)^2 + 1 \leq P(x) \leq 2(x-1)^2 + 1.$$

The left hand and right hand expressions represent parabolas with a vertex at $(1, 1)$, so $P(x)$ must also represent a parabola with vertex at $(1, 1)$. Therefore $P(x) = a(x-1)^2+1$, $P(11) = 100a+1 = 181$, and $a = \frac{9}{5}$. Thus $P(x) = \frac{9}{5}(x-1)^2 + 1$, and $P(16) = 406$.

7. (Answer: 760)

Let $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7\}$. If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a minimally intersecting ordered triple of subsets of $\mathcal{S}$, then there exist distinct $x, y, z \in \mathcal{S}$ such that $\mathcal{A} \cap \mathcal{B} = \{x\}$, $\mathcal{B} \cap \mathcal{C} = \{y\}$, and $\mathcal{A} \cap \mathcal{C} = \{z\}$. There are $7 \cdot 6 \cdot 5$ ways to assign values to x, y, z. For each of the four remaining elements of $\mathcal{S}$, there are four possibilities, namely that either the element belongs to exactly one of the three sets $\mathcal{A}, \mathcal{B}$, or $\mathcal{C}$, or it belongs to none of the three sets. Thus there are a total of $7 \cdot 6 \cdot 5 \cdot 4^4$ minimally intersecting ordered triples of sets in which each set in each triple is a subset of $\mathcal{S}$. Thus $N = 7 \cdot 6 \cdot 5 \cdot 4^4 = 210 \cdot 256 = 53760$, and the requested remainder is 760.

8. (Answer: 132)



Suppose $(x, y) \in \mathcal{R}$. Because $\lfloor a \rfloor$ takes integer values and $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25$, the ordered pairs $(\lfloor x \rfloor, \lfloor y \rfloor)$ must be elements of the set

$$S = \{(\pm 5, 0), (\pm 4, \pm 3), (\pm 3, \pm 4), (0, \pm 5)\}.$$

Thus region $\mathcal{R}$ is a subset of the 12 unit-square regions with lower left corners in S.

Region $\mathcal{R}$ is symmetric about point $Q = (\frac{1}{2}, \frac{1}{2})$. Points $A = (4,5)$, $B = (5,4)$, $C = (-3,-4)$, $D = (-4,-3)$ lie on the boundary of region $\mathcal{R}$. Thus $ABCD$ is a rectangle centered at Q. The smallest circle that can be drawn to cover these four points (which are boundary points of $\mathcal{R}$) is the circumcircle of $ABCD$, which has diameter $AC = \sqrt{7^2 + 9^2} = \sqrt{130}$. Substitution of the vertices of the twelve unit square regions into the inequality defining this circumcircle and its interior confirms that this circle does cover region $\mathcal{R}$. Hence the minimum value of r is $\frac{\sqrt{130}}{2}$, and $m + n = 132$.

9. (Answer: 158)

Add xyz to each side of each equation to obtain $x^3 = 2 + xyz$, $y^3 = 6 + xyz$, and $z^3 = 20 + xyz$. Letting $P = xyz$, it follows that $P^3 = (2+P)(6+P)(20+P) = P^3 + 28P^2 + 172P + 240$. Simplifying yields the equation $7P^2 + 43P + 60 = 0$, and thus $P = -\frac{15}{7}$ or $P = -4$. By adding the original three equations, it follows that $x^3 + y^3 + z^3 = 28 + 3P$, and this can be maximized by taking the greater of the two values of P, which is

$-\frac{15}{7}$. This value of P corresponds to the solution $\left(-\frac{1}{\sqrt[3]{7}}, \frac{3}{\sqrt[3]{7}}, \frac{5}{\sqrt[3]{7}}\right)$ of the system. Thus the largest possible value of $x^3 + y^3 + z^3$ is $28 - \frac{45}{7} = \frac{151}{7}$, and $m + n = 151 + 7 = 158$.

Note: The solution corresponding to $P = -4$ is $\left(-\sqrt[3]{2}, \sqrt[3]{2}, 2\sqrt[3]{2}\right)$.

10. (Answer: 202)

Write $a_i = 10b_i + c_i$, where $b_i,\ c_i \in \{0, 1, 2, \ldots, 7, 8, 9\}$; if b_i and c_i are chosen in this way, they determine a unique acceptable a_i.

Let $m = b_3 \cdot 10^3 + b_2 \cdot 10^2 + b_1 \cdot 10^1 + b_0 \cdot 10^0$, and $n = c_3 \cdot 10^3 + c_2 \cdot 10^2 + c_1 \cdot 10^1 + c_0 \cdot 10^0$, and write the representation as

$$\begin{aligned} 2010 &= (10b_3 + c_3)10^3 + (10b_2 + c_2)10^2 + (10b_1 + c_1)10^1 + (10b_0 + c_0)10^0 \\ &= 10m + n. \end{aligned}$$

The number of such representations is the number of ways to write 2010 as $10m + n$, where m and n are nonnegative integers. That is, $m \in \{0, 1, \ldots, 201\}$ and $n = 2010 - 10m$. Thus $N = 202$.

11. (Answer: 365)

The region $\mathcal{R}$ is a triangular region bounded by the lines $3y - x = 15$, $y = x + 2$, and $y = -x + 18$. The vertices of this triangle are $A = \left(\dfrac{9}{2}, \dfrac{13}{2}\right)$, $B = \left(\dfrac{39}{4}, \dfrac{33}{4}\right)$, and $C = (8, 10)$. Let D be the foot of the perpendicular from C to line AB. It can be verified that the coordinates of point D are $(8.7, 7.9)$, and hence D is between A and B. Thus the solid of revolution consists of two right circular cones with heights AD and BD, each having a base radius of CD. The desired volume is therefore $\dfrac{1}{3}\pi \cdot CD^2 \cdot AD + \dfrac{1}{3}\pi \cdot CD^2 \cdot BD = \frac{1}{3}\pi \cdot CD^2 \cdot AB$. Note that

$$\begin{aligned} AB &= \sqrt{\left(\frac{21}{4}\right)^2 + \left(\frac{7}{4}\right)^2} = \frac{7}{4}\sqrt{3^2 + 1^2} = \frac{7\sqrt{10}}{4} \quad \text{and} \\ CD &= \sqrt{(8 - 8.7)^2 + (10 - 7.9)^2} = \frac{7}{\sqrt{10}}. \end{aligned}$$

Thus the desired volume is $\dfrac{1}{3}\pi \cdot \dfrac{49}{10} \cdot \dfrac{7\sqrt{10}}{4} = \dfrac{343\pi}{12\sqrt{10}}$, and $m + n + p = 343 + 12 + 10 = 365$.

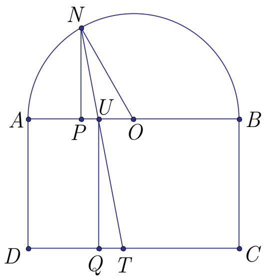
12. (Answer: 243)

First prove that $m \le 243$. Let $S = \{3, 4, \ldots, 243\}$ and assume that T and U form a partition of S such that neither of the subsets contains a solution to the given equation. Without loss of generality assume that $3 \in T$. Then necessarily $3^2 \in U$.

If $3^3 \in T$, then because $3^4 = 3 \cdot 3^3 = 3^2 \cdot 3^2$, one of the subsets must contain a solution to the given equation, which contradicts the assumption.

On the other hand, if $3^3 \in U$ then $3^4 \in T$ and thus $3^5 \in U$. This implies that $3^2 \in U$. Then the set U contains a solution to the given equation, which again contradicts the assumption. Thus no such partition exists. If $m \leq 242$, then $S = \{3, 4, \ldots, m\}$ can be partitioned by taking $T = \{3, 4, \ldots, 8, 81, 82, \ldots, m\}$, and $U = \{9, 10, \ldots, 80\}$, and neither of the sets contains a solution to the equation.

13. (Answer: 069)



Let O be the midpoint of segment $\overline{AB}$. Then $AO = (AU + UB)/2 = 126 = AN$ and $UO = 42$. Thus $\triangle AON$ is equilateral and $\angle AON = 60°$, implying that the ratio of the areas of sectors AON and NOB is $1:2$. Let Q be the foot of the perpendicular from U to line DC. Because $AU/UB = 1/2$, the ratio of the areas of rectangles $AUQD$ and $UQCB$ is also $1:2$. Because line l divides region $\mathcal{R}$ into two parts with area ratio $1:2$, it follows that triangles NUO and UQT have the same area. Let P be the foot of the perpendicular from N to line AB. Note that triangles NUP and UTQ are similar, with area ratio equal to

$$\frac{UQ^2}{NP^2} = \frac{\text{Area}(UTQ)}{\text{Area}(NUP)} = \frac{\text{Area}(NOU)}{\text{Area}(NUP)} = \frac{UO}{UP} \text{ or } UQ = NP \cdot \sqrt{\frac{UO}{UP}}.$$

In $\triangle NOP$, $NO = AO = 126$, $\angle NPO = 90°$, and $\angle NOP = 60°$. Hence $NP = 63\sqrt{3}$, $OP = 63$, $UP = OP - UO = 21$, and $\sqrt{\frac{UO}{UP}} = \sqrt{\frac{42}{21}} = \sqrt{2}$. Therefore

$$DA = UQ = NP \cdot \sqrt{\frac{UO}{UP}} = 63\sqrt{6},$$

and $m + n = 69$.

14. (Answer: 109)

Note that $f(1) = 9 \cdot 0 + 90 \cdot 1 + 2 = 92$, and $f(10n) = 100 + f(n)$, so $f(100) = 292$ and $f(1000) = 392$. For $0 \leq j < 900$, $\log_{10}(k(100+j)) \geq 2$. Furthermore $\log_{10}(k(100+j)) \geq 3$ if and only if

$$k \geq \frac{1000}{100+j} = 10 - \frac{10j}{100+j}, \text{ that is, } k \geq 10 - \left\lfloor \frac{10j}{100+j} \right\rfloor.$$

Therefore the number of terms in the sequence having a value of at least 3 is $91 + \left\lfloor \frac{10j}{100+j} \right\rfloor$. Similarly, the number of terms having a value of 4 is $1 + \left\lfloor \frac{100j}{100+j} \right\rfloor$, which implies

$$f(100+j) = 200 + 91 + 1 + \left\lfloor \frac{10j}{100+j} \right\rfloor + \left\lfloor \frac{100j}{100+j} \right\rfloor.$$

Thus the required value of $n = 100 + j$ must satisfy $100j < 9(100+j)$, and therefore $j \leq 9$. It can be verified that $f(109) = 300$, so the answer is 109.

15. (Answer: 045)

Let $\frac{AM}{CM} = k$ and the common altitude of $\triangle AMB$ and $\triangle CMB$ be h. Because the radius of the incircle of triangle equals twice its area divided by its perimeter, the ratio of the areas of two triangles with equal inradii is the same as the ratio of their perimeters. Thus $\frac{12 + AM + BM}{13 + CM + BM} = \frac{\frac{1}{2}AM \cdot h}{\frac{1}{2}CM \cdot h} = k$. Replacing AM by $k \cdot CM$ and solving for BM yields $BM = \frac{13k - 12}{1-k}$. The fact that $BM > 0$ implies that $\frac{12}{13} < k$. Because $\frac{AM}{CM} = k$ and $AM + CM = 15$, it follows that $CM = \frac{15}{k+1}$ and $AM = \frac{15k}{k+1}$. Applying the Law of Cosines to triangles ABM and BCM and to angles $\angle BMA = \alpha$ and $\angle CMB = \pi - \alpha$ respectively yields

$$12^2 = AM^2 + BM^2 - 2AM \cdot BM \cos\alpha, \text{ and}$$
$$13^2 = BM^2 + CM^2 + 2BM \cdot CM \cos\alpha.$$

Using $AM = k \cdot CM$, multiplying the second equation by k, and adding the two equations yields

$$13^2 k + 12^2 = BM^2(k+1) + AM^2 + CM^2 k.$$

Substituting into the above equation produces

$$169k + 144 = \left(\frac{13k - 12}{1 - k}\right)^2 (k + 1) + \left(\frac{15k}{k + 1}\right)^2 + \left(\frac{15}{k + 1}\right)^2 k.$$

Simplifying this equation yields $4k(69k^2 - 112k + 44) = 0$. Its solutions are $k = 0$, $k = \dfrac{2}{3}$, and $k = \dfrac{22}{23}$. Because $k > \frac{12}{13}$, only the last solution is valid, and so $p + q = 45$.