1. **Answer (C):** Distributing the negative signs gives

$$\begin{aligned}&(20-(2010-201))+(2010-(201-20))\\&=20-2010+201+2010-201+20\\&=40.\end{aligned}$$

2. **Answer (A):** The ferry boat makes 6 trips to the island. The number of tourists shuttled was

$$\begin{aligned}&100+(100-1)+(100-2)+(100-3)+(100-4)+(100-5)\\&=6\cdot 100-(1+2+3+4+5)\\&=600-15\\&=585.\end{aligned}$$

3. **Answer (E):** Let s equal the side length of the square. Because half of the area of the rectangle is in the square, $\frac{1}{2}AB=s$. Because one fifth of the square's area is in the shaded region, $s=5\cdot AD$. Therefore $\frac{1}{2}AB=5\cdot AD$, and $\frac{AB}{AD}=10$.

4. **Answer (D):** Choice **(D)** may be written as $-\frac{1}{x}$. If x is negative, choice **(D)** is positive. To see that the other choices need not be positive, let $x=-1$ and then

$$\begin{aligned}&\textbf{(A)}\,\frac{-1}{|-1|}=-1,\\&\textbf{(B)}\,-(-1)^2=-1,\\&\textbf{(C)}\,-2^{-1}=-\frac{1}{2},\\&\textbf{(E)}\,\sqrt[3]{-1}=-1.\end{aligned}$$

5. **Answer (C):** The second place archer could score a maximum of $50\cdot 10=500$ points with the remaining shots. Therefore Chelsea needs to score more than $500-50=450$ points to guarantee victory. If Chelsea's next n shots will score $10n$ points, her remaining $50-n$ shots will score at least $4(50-n)$ points. To guarantee victory,

$$\begin{aligned}10\cdot n+4\cdot(50-n)&>450\end{aligned}$$

$$\begin{aligned}6n+200&>450\\n&>41\frac{2}{3}.\end{aligned}$$

Therefore Chelsea needs at least 42 bullseyes to guarantee victory.

OR

If Chelsea does not make a bullseye, the maximum number of points her opponents could gain per shot would be $10-4=6$. Chelsea must make enough bullseyes to prevent her opponents from gaining 50 points. Because $8\cdot 6<50<9\cdot 6$, the most non-bullseyes she can afford to score is 8, leaving $50-8=42$ bullseyes needed to guarantee her victory.

6. **Answer (E):** Let $x+32$ be written in the form $CDDC$. Because x has three digits, $1000<x+32<1032$, and so $C=1$ and $D=0$. Hence $x=1001-32=969$, and the sum of the digits of x is $9+6+9=24$.

7. **Answer (C):** The volume scale for Logan's model is $0.1:100{,}000=1:1{,}000{,}000$. Therefore the linear scale is $1:\sqrt[3]{1{,}000{,}000}$, which is $1:100$. Logan's water tower should stand $\frac{40}{100}=0.4$ meters tall.

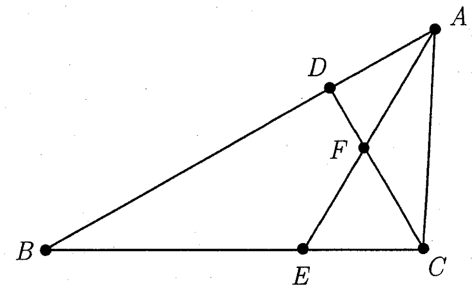
8. **Answer (C):** Let $\alpha=\angle BAE=\angle ACD=\angle ACF$. Because $\triangle CFE$ is equilateral, it follows that $\angle CFA=120^\circ$ and then

$$\angle FAC=180^\circ-120^\circ-\angle ACF=60^\circ-\alpha.$$

Therefore

$$\angle BAC=\angle BAE+\angle FAC=\alpha+(60^\circ-\alpha)=60^\circ.$$

Because $AB=2\cdot AC$, it follows that $\triangle BAC$ is a $30-60-90^\circ$ triangle, and thus $\angle ACB=90^\circ$.



9. **Answer (A):** The volume of the solid cube is 27 in^3. The first hole to be cut removes $2 \times 2 \times 3 = 12$ in^3 from the volume. The other holes remove $2 \times 2 \times 0.5 = 2$ in^3 from each of the four remaining faces. The volume of the remaining solid is $27 - 12 - 4(2) = 7$ in^3.

10. **Answer (A):** Consecutive terms in an arithmetic sequence have a common difference d. Thus $(3p + q) - (3p - q) = 2q = d$. Further, the second term is equal to $p + d$, so $p + d = 9$, and the third term is equal to $p + 2d$, so $p + 2d = 3p - q$. These three equations form a system that can be solved to yield $p = 5$, $q = 2$, and $d = 4$. Therefore the 2010^{th} term of the sequence is $p + 2009d = 5 + 2009 \cdot 4 = 8041$.

11. **Answer (C):** If $x = \log_b 7^7$, then $b^x = 7^7$. Thus

$$(7b)^x = 7^x \cdot b^x = 7^{x+7} = 8^x.$$

Because $x > 0$, it follows that $7b = 8$ and so $b = \frac{8}{7}$.

OR

Taking the logarithm of both sides gives us $(x + 7)\log 7 = x \log 8$. Solving, we have $\frac{x+7}{x} = \frac{\log 8}{\log 7}$, $x \log 8 = x \log 7 + 7 \log 7$, $x(\log 8 - \log 7) = 7 \log 7$, and we have $x = \dfrac{\log 7^7}{\log \frac{8}{7}}$. Using the change of base rule for logarithms, $b = \frac{8}{7}$.

12. **Answer (D):** LeRoy and Chris cannot both be frogs, because their statements would be true and frogs lie. Also LeRoy and Chris cannot both be toads, because then their statements would be false, and toads tell the truth. Hence between LeRoy and Chris, exactly one must be a toad.

If Brian is a toad, then Mike must be a frog, but this is a contradiction as Mike's statement would then be true. Hence Brian is a frog, so Brian's statement must be false, and Mike must be a frog. Altogether there are 3 frogs: Brian, Mike, and either LeRoy or Chris.

13. **Answer (C):** When $k = 0$, the graphs of $x^2 + y^2 = 0$ and $xy = 0$ consist of the single point $\{(0, 0)\}$ and the union of the two lines $x = 0$ and $y = 0$, respectively; so the two graphs intersect. When $k \neq 0$, the graph of $x^2 + y^2 = k^2$ is a circle of radius k centered at the origin and the graph of $xy = k$ is an equilateral hyperbola centered at the origin. The vertices of the hyperbola, located at

$(\pm\sqrt{k}, \pm\sqrt{k})$ if $k > 0$ or at $(\pm\sqrt{-k}, \mp\sqrt{-k})$ if $k < 0$, are the closest points on the graph to the origin. If $|k| \geq 2$, then

$$(\sqrt{|k|})^2 + (\sqrt{|k|})^2 = 2\,|k| \leq k^2,$$

thus the graphs intersect. If $|k| = 1$, then

$$(\sqrt{|k|})^2 + (\sqrt{|k|})^2 = 2 > 1 = k^2,$$

and thus the graphs do not intersect. Thus the graphs do not intersect for $k = 1$ or $k = -1$.

14. **Answer (B):** By the Angle Bisector Theorem, $8 \cdot BA = 3 \cdot BC$. Thus BA must be a multiple of 3. If $BA = 3$, the triangle is degenerate. If $BA = 6$, then $BC = 16$, and the perimeter is $6 + 16 + 11 = 33$.

15. **Answer (D):** Let p be the requested probability. If the coin is flipped four times, the probability of heads and tails appearing twice is $\binom{4}{2}p^2(1-p)^2 = \frac{1}{6}$, and because $0 \leq p \leq 1$ it follows that $p(1-p) = \frac{1}{6}$. Solving for p yields $p = \frac{1}{6}(3 \pm \sqrt{3})$ and because $p < 1/2$, the answer is $p = \frac{1}{6}(3 - \sqrt{3})$.

16. **Answer (B):** The probability that Bernardo picks a 9 is $\frac{3}{9} = \frac{1}{3}$. In this case, his three-digit number will begin with a 9 and will be larger than Silvia's three-digit number.

If Bernardo does not pick a 9, then Bernardo and Silvia will form the same number with probability

$$\frac{1}{\binom{8}{3}} = \frac{1}{56}.$$

If they do not form the same number then Bernardo's number will be larger $\frac{1}{2}$ of the time.

Hence the probability is

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2}(1 - \frac{1}{56}) = \frac{111}{168} = \frac{37}{56}.$$

17. **Answer (E):** Triangles ABC, CDE and EFA are congruent, so $\triangle ACE$ is equilateral. Let X be the intersection of the lines AB and EF and define Y and Z similarly as shown in the figure. Because $ABCDEF$ is equiangular, it

follows that $\angle XAF = \angle AFX = 60°$. Thus $\triangle XAF$ is equilateral. Let H be the midpoint of $\overline{XF}$. By the Pythagorean Theorem,

$$AE^2 = AH^2 + HE^2 = (\frac{\sqrt{3}}{2}r)^2 + (\frac{r}{2}+1)^2 = r^2 + r + 1$$

Thus, the area of $\triangle ACE$ is

$$\frac{\sqrt{3}}{4}AE^2 = \frac{\sqrt{3}}{4}(r^2 + r + 1).$$

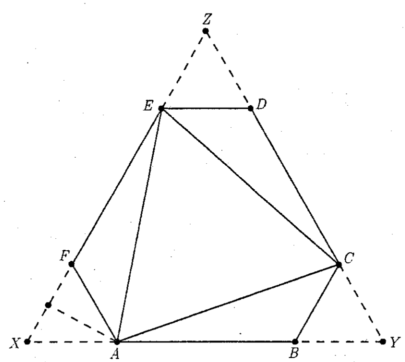
The area of hexagon $ABCDEF$ is equal to

$$[XYZ] - [XAF] - [YCB] - [ZED] = \frac{\sqrt{3}}{4}\left((2r+1)^2 - 3r^2\right) = \frac{\sqrt{3}}{4}(r^2 + 4r + 1)$$

Because $[ACE] = \frac{7}{10}[ABCDEF]$, it follows that

$$r^2 + r + 1 = \frac{7}{10}(r^2 + 4r + 1)$$

from which $r^2 - 6r + 1 = 0$ and $r = 3 \pm 2\sqrt{2}$. The sum of all possible values of r is 6.



18. **Answer (D):** Each such path intersects the line $y = -x$ at exactly one of the points $(\pm 4, \mp 4)$, $(\pm 3, \mp 3)$, or $(\pm 2, \mp 2)$. For $j = 0$, 1, and 2, the number of paths from $(-4, 4)$ to either of $(\pm(4-j), \mp(4-j))$ is $\binom{8}{j}$, and the number of paths to $(4, 4)$ from either of $(\pm(4-j), \mp(4-j))$ is the same. Therefore the number of paths that meet the requirement is $2\left(\binom{8}{0}^2 + \binom{8}{1}^2 + \binom{8}{2}^2\right) = 2(1^2 + 8^2 + 28^2) = 1698$.

19. **Answer (A):** If Isabella reaches the k^{th} box, she will draw a white marble from it with probability $\frac{k}{k+1}$. For $n \ge 2$, the probability that she will draw white marbles from each of the first $n-1$ boxes is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n},$$

so the probability that she will draw her first red marble from the n^{th} box is $P(n) = \frac{1}{n(n+1)}$. The condition $P(n) < 1/2010$ is equivalent to $n^2 + n - 2010 > 0$, from which $n > \frac{1}{2}(-1 + \sqrt{8041})$ and $(2n+1)^2 > 8041$. The smallest positive odd integer whose square exceeds 8041 is 91, and the corresponding value of n is 45.

20. **Answer (C):** Because $a_n = 1 + (n-1)d_1$ and $b_n = 1 + (n-1)d_2$ for some integers d_1 and d_2, it follows that $n-1$ is a factor of $\gcd(a_n - 1, b_n - 1)$. The ordered pair (a_n, b_n) must be one of $(2, 1005)$, $(3, 670)$, $(5, 402)$, $(6, 335)$, $(10, 201)$, $(15, 134)$, or $(30, 67)$. For every pair except the sixth pair, the numbers $a_n - 1$ and $b_n - 1$ are relatively prime, so $n = 2$. In the exceptional case, $\gcd(15-1, 134-1) = 7$. The sequences defined by $a_n = 2n - 1$ and $b_n = 19n - 18$ satisfy the conditions, so $n = 8$.

21. **Answer (A):** Let the three points of intersections have x-coordinates p, q, and r, and let $f(x) = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c$. Then $f(p) = f(q) = f(r) = 0$, and $f(x) \ge 0$ for all x, so $f(x) = \left((x-p)(x-q)(x-r)\right)^2 = \left(x^3 - Ax^2 + Bx - C\right)^2$, where $A = p+q+r$, $B = pq+qr+rp$, and $C = pqr$. The coefficient of x^5 is $-10 = -2A$, so $A = 5$. The coefficient of x^4 is $29 = A^2 + 2B = 25 + 2B$, so $B = 2$. The coefficient of x^3 is $-4 = -2C - 2AB = -2C - 20$, so $C = -8$. Thus $f(x) = \left(x^3 - 5x^2 + 2x + 8\right)^2$. Because the sums of the coefficients of the even and odd powers of x are equal, -1 is a zero of $f(x)$. Factoring gives $f(x) = \left((x+1)(x^2 - 6x + 8)\right)^2 = \left((x+1)(x-2)(x-4)\right)^2$, and the largest of the three zeros is 4.

22. **Answer (A):** Note that

$$f(x) = \begin{cases} -(x-1) - (2x-1) - \cdots - (119x-1), & \text{if } x \le \frac{1}{119}; \\ -(x-1) - (2x-1) - \cdots - ((m-1)x-1) & \\ \quad +(mx-1) + \cdots + (119x-1), & \text{if } \frac{1}{m} \le x \le \frac{1}{m-1};\ 2 \le m \le \\ (x-1) + (2x-1) + \cdots + (119x-1), & \text{if } x \ge 1. \end{cases}$$

The graph of $f(x)$ consists of a negatively sloped ray for $x \le \frac{1}{119}$, a positively sloped ray for $x \ge 1$, and for $\frac{1}{119} \le x \le 1$ a sequence of line segments whose slopes increase as x increases. The minimum value of $f(x)$ occurs at the right endpoint of the rightmost interval in which the graph has a non-positive slope. The slope on the interval $[\frac{1}{m}, \frac{1}{m-1}]$ is

$$\sum_{k=m}^{119} k - \sum_{k=1}^{m-1} k = \sum_{k=1}^{119} k - 2\sum_{k=1}^{m-1} k = 7140 - (m-1)(m).$$

The inequality $7140 + m - m^2 \leq 0$ is satisfied in the interval $[-84, 85]$ with equality at the endpoints. Therefore on the interval $[\frac{1}{85}, \frac{1}{84}]$ the graph of $f(x)$ has a slope of 0 and a constant value of $(84)(1) + (119 - 84)(-1) = 49$.

23. **Answer (A):** There are 18 factors of 90! that are multiples of 5, 3 factors that are multiples of 25, and no factors that are multiples of higher powers of 5. Also, there are more than 45 factors of 2 in 90!. Thus $90! = 10^{21}N$ where N is an integer not divisible by 10, and if $N \equiv n \pmod{100}$ with $0 < n \leq 99$, then n is a multiple of 4.

Let $90! = AB$ where A consists of the factors that are relatively prime to 5 and B consists of the factors that are divisible by 5. Note that $\prod_{j=1}^{4}(5k + j) \equiv 5k(1 + 2 + 3 + 4) + 1 \cdot 2 \cdot 3 \cdot 4 \equiv 24 \pmod{25}$, thus

$$\begin{aligned} A &= (1 \cdot 2 \cdot 3 \cdot 4) \cdot (6 \cdot 7 \cdot 8 \cdot 9) \cdot \cdots \cdot (86 \cdot 87 \cdot 88 \cdot 89) \\ &\equiv 24^{18} \equiv (-1)^{18} \equiv 1 \pmod{25}. \end{aligned}$$

Similarly,

$$B = (5 \cdot 10 \cdot 15 \cdot 20) \cdot (30 \cdot 35 \cdot 40 \cdot 45) \cdot (55 \cdot 60 \cdot 65 \cdot 70) \cdot (80 \cdot 85 \cdot 90) \cdot (25 \cdot 50 \cdot 75),$$

thus

$$\begin{aligned} \frac{B}{5^{21}} &= (1 \cdot 2 \cdot 3 \cdot 4) \cdot (6 \cdot 7 \cdot 8 \cdot 9) \cdot (11 \cdot 12 \cdot 13 \cdot 14) \cdot (16 \cdot 17 \cdot 18) \cdot (1 \cdot 2 \cdot 3) \\ &\equiv 24^3 \cdot (-9) \cdot (-8) \cdot (-7) \cdot 6 \equiv (-1)^3 \cdot 1 \equiv -1 \pmod{25}. \end{aligned}$$

Finally, $2^{21} = 2 \cdot (2^{10})^2 = 2 \cdot (1024)^2 \equiv 2 \cdot (-1)^2 \equiv 2 \pmod{25}$, so $13 \cdot 2^{21} \equiv 13 \cdot 2 \equiv 1 \pmod{25}$. Therefore

$$\begin{aligned} N &\equiv (13 \cdot 2^{21})N = 13 \cdot \frac{90!}{5^{21}} = 13 \cdot A \cdot \frac{B}{5^{21}} \equiv 13 \cdot 1 \cdot (-1) \pmod{25} \\ &\equiv -13 \equiv 12 \pmod{25}. \end{aligned}$$

Thus n is equal to 12, 37, 62, or 87, and because n is a multiple of 4, it follows that $n = 12$.

24. **Answer (B):**

Let $g(x) = \sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x)$. The domain of $f(x)$ is the union of all intervals on which $g(x) > 0$. Note that $\sin(n\pi(1x)) = (-1)^{k+1}\sin(n\pi x)$,

so $g(1x) = g(x)$. Because $g(1/2) = 0$, it suffices to consider the subintervals of $(0, 1/2)$ on which $g(x) > 0$. In this interval the distinct solutions of the equation $g(x) = 0$ are the numbers k/n, where $2 \leq n \leq 8$, $1 \leq k < n/2$, and k and n are relatively prime. For $n = 2, 3, 4, 5, 6, 7,$ and 8 there are, respectively, 0, 1, 1, 2, 1, 3, and 2 values of k. Thus there are $1 + 1 + 2 + 1 + 3 + 2 = 10$ solutions of $g(x) = 0$ in the interval $(0, 1/2)$. The sign of $g(x)$ changes at k/n unless an even number of factors of $g(x)$ are zero at k/n, that is unless there are an even number of ways to represent k/n as a rational number with a positive denominator not exceeding 8. Thus the sign of $g(n)$ changes except at $1/4 = 2/8$ and $1/3 = 2/6$.

Let the solutions of $g(x) = 0$ in the interval $(0, 1/2)$ be $x_1, x_2, \ldots, x_{10}$ in increasing order, and let $x_0 = 0$ and $x_{11} = 1/2$. It is easily verified that $x_5 = 1/4$ and $x_7 = 1/3$, so for $0 \leq j \leq 10$, the sign of $g(x)$ changes at x_j except for $j = 5$ and 7. Because 5 and 7 have the same parity and $g(x) > 0$ in (x_0, x_1), the solution of $g(x) > 0$ in $(0, 1/2)$ consists of 6 disjoint open intervals. The solution of $g(x) > 0$ in $(1/2, 1)$ also consists of 6 disjoint open intervals, so the requested number of intervals is 12.

25. **Answer (C):** Suppose that a quadrilateral with sides $a \geq b \geq c \geq d$ and with perimeter 32 exists. By the triangle inequality $a < b + c + d = 32 - a$, so $a \leq 15$. Reciprocally, if (a, b, c, d) is a quadruple of positive integers whose sum equals 32, and whose maximum entry is $a \leq 15$, then $b + c + d = 32 - a \geq 17 > a$, so the triangle inequality is satisfied. This is the only condition required to guarantee the existence of a convex quadrilateral with given side lengths. Moreover, if the cyclic order of the sides is specified, then there is exactly one such cyclic quadrilateral.

The problem reduces to counting all the quadruples (a, b, c, d) of positive integers with $a + b + c + d = 32$, $\max(a, b, c, d) \leq 15$, and where two quadruples are considered the same if they generate the same quadrilateral, that is if one is a cyclic permutation of the other one. For example $(12, 4, 5, 11)$ and $(5, 11, 12, 4)$ generate the same quadrilateral.

The number of quadruples (a, b, c, d) with $a + b + c + d = 32$ can be counted as follows: consider 31 spots on a line to be filled with 28 ones and 3 plus signs. There are $\binom{31}{3}$ ways to choose the locations of the plus signs, and every such assignment is in one-to-one correspondence to the quadruple (a', b', c', d'), where each entry indicates the number of ones between consecutive plus signs. Setting $(a, b, c, d) = (1, 1, 1, 1) + (a', b', c', d')$ gives precisely all quadruples where $a, b, c, d \geq 1$ and $a + b + c + d = 32$. To count those where the maximum entry is 16 or more, consider 13 ones and 3 plus signs. There are $\binom{16}{3}$ quadruples (a', b', c', d') where $a', b', c', d' \geq 0$ and $a' + b' + c' + d' = 13$, there are 4 ways to choose one of the coordinates, say a', to be the maximum. Then the quadruple $(a, b, c, d) = (16, 1, 1, 1) + (a', b', c', d')$ satisfies our requirements. Thus there are exactly $4\binom{16}{3}$ quadruples (a, b, c, d) where $a, b, c, d \geq 1$, $a + b + c + d = 32$, and

$\max(a, b, c, d) \geq 16$; consequently, there are

$$\binom{31}{3} - 4\binom{16}{3} \tag{1}$$

quadruples (a, b, c, d) where $a, b, c, d \geq 1$, $a+b+c+d = 32$, and $\max(a, b, c, d) \leq 15$.

If (a, b, c, d) consists of distinct entries, then it has exactly 4 cyclic permutations. The same occurs if only two entries are equal to each other, or three entries are equal to each other and the remaining entry is not. If (a, b, c, d) has two pairs of entries equal to each other ordered (a, a, b, b), then it has 4 cyclic permutations, but if they are ordered (a, b, a, b) then it has only 2 cyclic permutations. Finally, if all entries are equal then there is only one cyclic permutation.

There are exactly $2 \cdot 7 = 14$ quadruples of the form (a, b, a, b) with $a \neq b$ and $a + b = 16$ and there is only one quadruple $(a, a, a, a) = (8, 8, 8, 8)$ with four equal entries. Adding to (1) the number of quadruples of the form (a, b, a, b) and 3 times the number of quadruples of the form (a, a, a, a), guarantees that all classes of equivalence under cyclic permutations are counted exactly 4 times. Therefore the required number of cyclic quadrilaterals is

$$\begin{aligned} \frac{1}{4}\left(\binom{31}{3} - 4\binom{16}{3} + 14 + 3\right) &= \frac{1}{4}(31 \cdot 5 \cdot 29 - 32 \cdot 5 \cdot 14 + 17) \\ &= \frac{1}{4}(5 \cdot 451 + 17) = 568. \end{aligned}$$