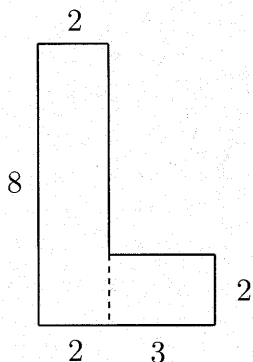


1. **Answer (C):** Makayla spent $45 + 2 \cdot 45 = 135$ minutes, or $\frac{135}{60} = \frac{9}{4}$ hours in meetings. Hence she spent $100 \cdot \frac{9/4}{9} = 25$ percent of her time in meetings.
2. **Answer (A):** The region consists of two rectangles: an 8-by-2 rectangle, and a 3-by-2 rectangle. The desired area is $8 \cdot 2 + 3 \cdot 2 = 22$.



3. **Answer (E):** The cost of an individual ticket must divide 48 and 64. The common factors of 48 and 64 are 1, 2, 4, 8, and 16. Each of these may be the cost of one ticket, so there are 5 possible values for x .
4. **Answer (B):** A month with 31 days has 3 successive days of the week appearing five times and 4 successive days of the week appearing four times. If Monday and Wednesday appear five times then Monday must be the first day of the month. If Monday and Wednesday appear only four times then either Thursday or Friday must be the first day of the month. Hence there are 3 days of the week that could be the first day of the month.
5. **Answer (D):** The correct answer was $1 - (2 - (3 - (4 + e))) = 1 - 2 + 3 - 4 - e = -2 - e$. Larry's answer was $1 - 2 - 3 - 4 + e = -8 + e$. Therefore $-2 - e = -8 + e$, so $e = 3$.
6. **Answer (D):** Assume there are 100 students in Mr. Wells' class. Then at least $70 - 50 = 20$ students answered "No" at the beginning of the school year and "Yes" at the end, so $x \geq 20$. Because only 30 students answered "No" at the end of the school year, at least $50 - 30 = 20$ students who answered "Yes" at the

beginning of the year gave the same answer at the end, so $x \leq 80$. The difference between the maximum and minimum possible values of x is $80 - 20 = 60$. The minimum $x = 20$ is achieved if exactly 20 students answered "No" at the beginning and "Yes" at the end of the school year. The maximum $x = 80$ is achieved if exactly 20 students answered "Yes" at the beginning and the end.

7. **Answer (C):** Let t be the number of minutes Shelby spent driving in the rain. Then she traveled $20 \frac{t}{60}$ miles in the rain, and $30 \frac{40-t}{60}$ miles in the sun. Solving $20 \frac{t}{60} + 30 \frac{40-t}{60} = 16$ results in $t = 24$ minutes.
8. **Answer (B):** If there are n schools in the city, then there are $3n$ contestants, so $3n \geq 64$, and $n \geq 22$. Because Andrea received the median score and each student received a different score, n is odd, so $n \geq 23$. Andrea's position is $\frac{3n+1}{2}$, and Andrea finished ahead of Beth, so $\frac{3n+1}{2} < 37$, and $3n < 73$. Because n is an odd integer, $n \leq 23$. Therefore $n = 23$.
9. **Answer (E):** Because n is divisible by 20, $n = 2^{2+a} \cdot 5^{1+b} \cdot k$, where a and b are nonnegative integers and k is a positive integer not divisible by 2 or 5. Because $n^2 = 2^{2(2+a)} \cdot 5^{2(1+b)} \cdot k^2$ is a perfect cube, 3 divides $2(2+a)$ and 3 divides $2(1+b)$. Because $n^3 = 2^{3(2+a)} \cdot 5^{3(1+b)} \cdot k^3$ is a perfect square, 2 divides $3(2+a)$ and 2 divides $3(1+b)$. Therefore 6 divides $2+a$ and 6 divides $1+b$. The smallest possible choices for a, b , and k , are $a = 4, b = 5$, and $k = 1$. In this case $n = 2^6 \cdot 5^6 = 1,000,000$, and n has 7 digits.

OR

The only prime factors of 20 are 2 and 5, so n has the form $2^a \cdot 5^b$ for integers $a \geq 2$ and $b \geq 1$. Because n^2 is a perfect cube, $2a$ and $2b$ are both multiples of 3, so a and b are also both multiples of 3. Similarly, because n^3 is a perfect square, a and b are both multiples of 2. Therefore both a and b are multiples of 6. Note that $n = 2^6 \cdot 5^6 = 1,000,000$ satisfies the given conditions, and n has 7 digits.

10. **Answer (B):** The average of the numbers is "No"

$$\frac{1 + 2 + \cdots + 99 + x}{100} = \frac{99 \cdot 100}{2} + x}{100} = \frac{99 \cdot 50 + x}{100} = 100x.$$

This equation is equivalent to $9999x = (99 \cdot 101)x = 99 \cdot 50$, so $x = \frac{50}{101}$.

11. **Answer (E):** Each four-digit palindrome has digit representation $abba$ with $1 \leq a \leq 9$ and $0 \leq b \leq 9$. The value of the palindrome is $1001a + 110b$. Because 1001 is divisible by 7 and 110 is not, the palindrome is divisible by 7 if and only if $b = 0$ or $b = 7$. Thus the requested probability is $\frac{2}{10} = \frac{1}{5}$.

12. **Answer (D):** Rewriting each logarithm in base 2 gives

$$\frac{\frac{1}{2} \log_2 x}{\frac{1}{2}} + \log_2 x + \frac{2 \log_2 x}{2} + \frac{3 \log_2 x}{3} + \frac{4 \log_2 x}{4} = 40.$$

Therefore $5 \log_2 x = 40$, so $\log_2 x = 8$, and $x = 256$.

OR

For $a \neq 0$ the expression $\log_{2^a}(x^a) = y$ if and only if $2^{ay} = x^a$. Thus $2^y = x$ and $y = \log_2 x$. Therefore the given equation is equivalent to $5 \log_2 x = 40$, so $\log_2 x = 8$ and $x = 256$.

13. **Answer (C):** The maximum value for $\cos x$ and $\sin x$ is 1; hence $\cos(2A - B) = 1$ and $\sin(A + B) = 1$. Therefore $2A - B = 0^\circ$ and $A + B = 90^\circ$, and solving gives $A = 30^\circ$ and $B = 60^\circ$. Hence $\triangle ABC$ is a $30-60-90^\circ$ right triangle and $BC = 2$.

14. **Answer (B):** Note that $3M > (a + b) + c + (d + e) = 2010$, so $M > 670$. Because M is an integer $M \geq 671$. The value of 671 is achieved if $(a, b, c, d, e) = (669, 1, 670, 1, 669)$.

15. **Answer (D):** There are three cases to consider.

First, suppose that $i^x = (1+i)^y \neq z$. Note that $|i^x| = 1$ for all x , and $|(1+i)^y| \geq |1+i| = \sqrt{2} > 1$ for $y \geq 1$. If $y = 0$, then $(1+i)^y = 1 = i^x$ if x is a multiple of 4. The ordered triples that satisfy this condition are $(4k, 0, z)$ for $0 \leq k \leq 4$ and $0 \leq z \leq 19$, $z \neq 1$. There are $5 \cdot 19 = 95$ such triples.

Next, suppose that $i^x = z \neq (1+i)^y$. The only nonnegative integer value of i^x is 1, which is assumed when $x = 4k$ for $0 \leq k \leq 4$. In this case $i^x = 1$ and $y \neq 0$. The ordered triples that satisfy this condition are $(4k, y, 1)$ for $0 \leq k \leq 4$ and $1 \leq y \leq 19$. There are $5 \cdot 19 = 95$ such triples.

Finally, suppose that $(1+i)^y = z \neq i^x$. Note that $(1+i)^2 = 2i$, so $(1+i)^y$ is a positive integer only when y is a multiple of 8. Because $(1+i)^0 = 1$, $(1+i)^8 = (2i)^4 = 16$, and $(1+i)^{16} = 16^2 = 256$, the only possible ordered

triples are $(x, 0, 1)$ with $x \neq 4k$ for $0 \leq k \leq 4$ and $(x, 8, 16)$ for any x . There are $15 + 20 = 35$ such triples.

The total number of ordered triples that satisfy the given conditions is $95 + 95 + 35 = 225$.

16. **Answer (E):** Let $N = abc + ab + a = a(bc + b + 1)$. If a is divisible by 3, then N is divisible by 3. Note that 2010 is divisible by 3, so the probability that a is divisible by 3 is $\frac{1}{3}$.

If a is not divisible by 3 then N is divisible by 3 if $bc + b + 1$ is divisible by 3. Define b_0 and b_1 so that $b = 3b_0 + b_1$ is an integer and b_1 is equal to 0, 1, or 2. Note that each possible value of b_1 is equally likely. Similarly define c_0 and c_1 . Then

$$\begin{aligned} bc + b + 1 &= (3b_0 + b_1)(3c_0 + c_1) + 3b_0 + b_1 + 1 \\ &= 3(3b_0c_0 + c_0b_1 + c_1b_0 + b_0) + b_1c_1 + b_1 + 1. \end{aligned}$$

Hence $bc + b + 1$ is divisible by 3 if and only if $b_1 = 1$ and $c_1 = 1$, or $b_1 = 2$ and $c_1 = 0$. The probability of this occurrence is $\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

Therefore the requested probability is $\frac{1}{3} + \frac{2}{9} = \frac{13}{27}$.

17. **Answer (D):** Let a_{ij} denote the entry in row i and column j . The given conditions imply that $a_{11} = 1$, $a_{33} = 9$, and $a_{22} = 4, 5$, or 6 . If $a_{22} = 4$, then $\{a_{12}, a_{21}\} = \{2, 3\}$, and the sets $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$ are complementary subsets of $\{5, 6, 7, 8\}$. There are $\binom{4}{2} = 6$ ways to choose $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$, and only one way to order the entries. There are 2 ways to order $\{a_{12}, a_{21}\}$, so 12 arrays with $a_{22} = 4$ meet the given conditions. Similarly, the conditions are met by 12 arrays with $a_{22} = 6$. If $a_{22} = 5$, then $\{a_{12}, a_{13}, a_{23}\}$ and $\{a_{21}, a_{31}, a_{32}\}$ are complementary subsets of $\{2, 3, 4, 6, 7, 8\}$ subject to the conditions $a_{12} < 5$, $a_{21} < 5$, $a_{32} > 5$, and $a_{23} > 5$. Thus $\{a_{12}, a_{13}, a_{23}\} \neq \{2, 3, 4\}$ or $\{6, 7, 8\}$, so its elements can be chosen in $\binom{6}{3} - 2 = 18$ ways. Both the remaining entries and the ordering of all entries are then determined, so 18 arrays with $a_{22} = 5$ meet the given conditions.

Altogether, the conditions are met by $12 + 12 + 18 = 42$ arrays.

18. **Answer (C):** Let A denote the frog's starting point, and let P , Q , and B denote its positions after the first, second, and third jumps, respectively. Introduce a coordinate system with P at $(0, 0)$, Q at $(1, 0)$, A at $(\cos \alpha, \sin \alpha)$, and B at $(1 + \cos \beta, \sin \beta)$. It may be assumed that $0 \leq \alpha \leq \pi$ and $0 \leq \beta \leq 2\pi$. For $\alpha = 0$, the required condition is met for all values of β . For $\alpha = \pi$, the required condition is met only if $\beta = \pi$. For $0 < \alpha < \pi$, $AB = 1$ if and only if $\beta = \alpha$ or $\beta = \pi$, and the required condition is met if and only if $\alpha \leq \beta \leq \pi$. In

the $\alpha\beta$ -plane, the rectangle $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq 2\pi$ has area $2\pi^2$. The triangle $0 \leq \alpha \leq \pi$, $\alpha \leq \beta \leq \pi$ has area $\frac{\pi^2}{2}$, so the requested probability is $\frac{1}{4}$.

19. **Answer (E):** The Raiders' score was $a(1+r+r^2+r^3)$, where a is a positive integer and $r > 1$. Because ar is also an integer, $r = m/n$ for relatively prime positive integers m and n with $m > n$. Moreover $ar^3 = a \cdot \frac{m^3}{n^3}$ is an integer, so n^3 divides a . Let $a = n^3A$. Then the Raiders' score was $R = A(n^3 + mn^2 + m^2n + m^3)$, and the Wildcats' score was $R-1 = a+(a+d)+(a+2d)+(a+3d) = 4a+6d$ for some positive integer d . Because $A \geq 1$, the condition $R \leq 100$ implies that $n \leq 2$ and $m \leq 4$. The only possibilities are $(m, n) = (4, 1), (3, 2), (3, 1)$, or $(2, 1)$. The corresponding values of R are, respectively, $85A, 65A, 40A$, and $15A$. In the first two cases $A = 1$, and the corresponding values of $R-1$ are, respectively, $64 = 32+6d$ and $84 = 4+6d$. In neither case is d an integer. In the third case $40A = 40a = 4a + 6d + 1$ which is impossible in integers. In the last case $15a = 4a + 6d + 1$, from which $11a = 6d + 1$. The only solution in positive integers for which $4a+6d \leq 100$ is $(a, d) = (5, 9)$. Thus $R = 5+10+20+40 = 75$, $R-1 = 5+14+23+32 = 74$, and the number of points scored in the first half was $5+10+5+14 = 34$.

20. **Answer (E):** The ratio between consecutive terms of the sequence is

$$r = \frac{a_2}{a_1} = \cot x,$$

so $a_4 = (\tan x)(\cot x) = 1$, and r is also equal to

$$\sqrt{\frac{a_4}{a_2}} = \frac{1}{\sqrt{\cos x}}.$$

Therefore x satisfies the equation $\cos^3 x = \sin^2 x = 1 - \cos^2 x$, which can be written as $(\cos^2 x)(1 + \cos x) = 1$. The given conditions imply that $\cos x \neq 0$, so this equation is equivalent to

$$1 + \cos x = \frac{1}{\cos^2 x} = r^4.$$

Thus $1 + \cos x = 1 \cdot r^4 = a_4 \cdot r^4 = a_8$.

21. **Answer (B):** Because 1, 3, 5, and 7 are roots of the polynomial $P(x) - a$, it follows that

$$P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. The previous identity must hold for $x = 2, 4, 6$, and 8, thus

$$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8).$$

Therefore $315 = \text{lcm}(15, 9, 105)$ divides a , that is a is an integer multiple of 315. Let $a = 315A$. Because $Q(2) = Q(6) = 42A$, it follows that $Q(x) - 42A = (x-2)(x-6)R(x)$ where $R(x)$ is a polynomial with integer coefficients. Because $Q(4) = -70A$ and $Q(8) = -6A$ it follows that $-112A = -4R(4)$ and $-48A = 12R(8)$, that is $R(4) = 28A$ and $R(8) = -4A$. Thus $R(x) = 28A + (x-4)(-6A + (x-8)T(x))$ where $T(x)$ is a polynomial with integer coefficients. Moreover, for any polynomial $T(x)$ and any integer A , the polynomial $P(x)$ constructed this way satisfies the required conditions. The required minimum is obtained when $A = 1$ and so $a = 315$.

22. **Answer (D):** Let R be the circumradius of $ABCD$ and let $a = AB$, $b = BC$, $c = CD$, $d = DA$, and $k = bc = ad$. Because the areas of $\triangle ABC$, $\triangle CDA$, $\triangle BCD$, and $\triangle ABD$ are

$$\frac{ab \cdot AC}{4R}, \quad \frac{cd \cdot AC}{4R}, \quad \frac{bc \cdot BD}{4R}, \quad \text{and} \quad \frac{ad \cdot BD}{4R},$$

respectively, and $\text{Area}(\triangle ABC) + \text{Area}(\triangle CDA) = \text{Area}(\triangle BCD) + \text{Area}(\triangle ABD)$, it follows that

$$\frac{AC}{4R}(ab + cd) = \frac{BD}{4R}(bc + ad) = \frac{BD}{4R}(2k);$$

that is, $(ab + cd) \cdot AC = 2k \cdot BD$. By Ptolemy's Theorem $ac + bd = AC \cdot BD$. Solving for AC and substituting into the previous equation gives

$$BD^2 = \frac{1}{2k}(ac + bd)(ab + cd) = \frac{1}{2k}(a^2k + c^2k + b^2k + d^2k) = \frac{1}{2}(a^2 + b^2 + c^2 + d^2).$$

None of the sides can be equal to 11 or 13 because by assumption a, b, c , and d are pairwise distinct and less than 15, and so it is impossible to have a factor of 11 or 13 on each side of the equation $bc = ad$. If the largest side length is 12 or less, then $2BD^2 \leq 12^2 + 10^2 + 9^2 + 8^2 = 389$, and so $BD \leq \sqrt{\frac{389}{2}}$. If the largest side is 14 and the other sides are $s_1 > s_2 > s_3$, then $14s_3 = s_1s_2$. Thus 7 divides s_1s_2 and because $0 < s_2 < s_1 < 14$, it follows that either $s_1 = 7$ or $s_2 = 7$. If $s_1 = 7$, then $2BD^2 < 14^2 + 7^2 + 6^2 + 5^2 = 306$. If $s_2 = 7$, then $2s_3 = s_1$, and it follows that $2BD^2 \leq 14^2 + 7^2 + 12^2 + 6^2 = 425$. Therefore $BD \leq \sqrt{\frac{425}{2}}$ with equality for a cyclic quadrilateral with $a = 14$, $b = 12$, $c = 7$, and $d = 6$.

23. **Answer (A):** Because both $P(Q(x))$ and $Q(P(x))$ have four distinct real zeros, both $P(x)$ and $Q(x)$ must have two distinct real zeros, so there are real numbers h_1, k_1, h_2 , and k_2 such that $P(x) = (x-h_1)^2 - k_1^2$ and $Q(x) = (x-h_2)^2 - k_2^2$. The zeros of $P(Q(x))$ occur when $Q(x) = h_1 \pm k_1$. The solutions of each equation are equidistant from h_2 , so $h_2 = -19$. It follows that $Q(-15) - Q(-17) = (16 - k_2^2) - (4 - k_2^2) = 12$, and also $Q(-15) - Q(-17) = 2k_1$, so $k_1 = 6$. Similarly $h_1 = -54$, so $2k_2 = P(-49) - P(-51) = (25 - k_1^2) - (9 - k_1^2) = 16$, and $k_2 = 8$. Thus the sum of the minimum values of $P(x)$ and $Q(x)$ is $-k_1^2 - k_2^2 = -100$.

24. **Answer (C):** Let $f(x) = \frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011}$. Note that

$$f(x) - f(y) = (y-x) \left(\frac{1}{(x-2009)(y-2009)} + \frac{1}{(x-2010)(y-2010)} + \frac{1}{(x-2011)(y-2011)} \right).$$

If $x < y < 2009$, then $y-x > 0$,

$$\frac{1}{(x-2009)(y-2009)} > 0, \quad \frac{1}{(x-2010)(y-2010)} > 0,$$

and $\frac{1}{(x-2011)(y-2011)} > 0$.

Thus f is decreasing on the interval $x < 2009$, and because $f(x) < 0$ for $x < 0$, it follows that no values $x < 2009$ satisfy $f(x) \geq 1$.

If $2009 < x < y < 2010$, then $f(x) - f(y) > 0$ as before. Thus f is decreasing in the interval $2009 < x < 2010$. Moreover, $f(2009 + \frac{1}{10}) = 10 - \frac{10}{9} - \frac{10}{19} > 1$ and $f(2010 - \frac{1}{10}) = \frac{10}{9} - 10 - \frac{10}{11} < 1$. Thus there is a number $2009 < x_1 < 2010$ such that $f(x) \geq 1$ for $2009 < x \leq x_1$ and $f(x) < 1$ for $x_1 < x < 2010$.

Similarly, f is decreasing on the interval $2010 < x < 2011$, $f(2010 + \frac{1}{10}) > 1$, and $f(2011 - \frac{1}{10}) < 1$. Thus there is a number $2010 < x_2 < 2011$ such that $f(x) \geq 1$ for $2010 < x \leq x_2$ and $f(x) < 1$ for $x_2 < x < 2011$.

Finally, f is decreasing on the interval $x > 2011$, $f(2011 + \frac{1}{10}) > 1$, and $f(2014) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} < 1$. Thus there is a number $x_3 > 2011$ such that $f(x) \geq 1$ for $2011 < x \leq x_3$ and $f(x) < 1$ for $x > x_3$.

The required sum of the lengths of these three intervals is

$$x_1 - 2009 + x_2 - 2010 + x_3 - 2011 = x_1 + x_2 + x_3 - 6020.$$

Multiplying both sides of the equation

$$\frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011} = 1$$

by $(x-2009)(x-2010)(x-2011)$ and collecting terms on one side of the equation gives

$$x^3 - x^2(2009 + 2010 + 2011 + 1 + 1 + 1) + ax + b = 0$$

where a and b are real numbers. The three roots of this equation are x_1 , x_2 , and x_3 . Thus $x_1 + x_2 + x_3 = 6020 + 3$, and consequently the required sum equals 3.

25. **Answer (D):** Observe that $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Let $P = \prod_{n=2}^{5300} \text{pow}(n) = 2^a \cdot 3^b \cdot 5^c \cdot 67^d \cdot Q$ where Q is relatively prime with 2, 3, 5, and 67. The largest power of 2010 that divides P is equal to 2010^m where $m = \min(a, b, c, d)$.

By definition $\text{pow}(n) = 2^k$ if and only if $n = 2^k$. Because $2^{12} = 4096 < 5300 < 8192 = 2^{13}$, it follows that

$$a = 1 + 2 + \cdots + 12 = \frac{12 \cdot 13}{2} = 78.$$

Similarly, $\text{pow}(n) = 67$ if and only if $n = 67N$ and the largest prime dividing N is smaller than 67. Because $5300 = 79 \cdot 67 + 7$ and 71, 73, and 79 are the only primes p in the range $67 < p \leq 79$; it follows that for $n \leq 5300$, $\text{pow}(n) = 67$ if and only if

$$n \in \{67k : 1 \leq k \leq 79\} \setminus \{67^2, 67 \cdot 71, 67 \cdot 73, 67 \cdot 79\}.$$

Because $67^2 < 5300 < 2 \cdot 67^2$, the only $n \leq 5300$ for which $\text{pow}(n) = 67^k$ with $k \geq 2$, is $n = 67^2$. Therefore

$$d = 79 - 4 + 2 = 77.$$

If $n = 2^j \cdot 3^k$ for $j \geq 0$ and $k \geq 1$, then $\text{pow}(n) = 3^k$. Moreover, if $0 \leq j \leq 2$ and $1 \leq k \leq 6$, or if $0 \leq j \leq 1$ and $k = 7$; then $n = 2^j \cdot 3^k \leq 2 \cdot 3^7 = 4374 < 5300$. Thus

$$b \geq 3(1 + 2 + \cdots + 6) + 7 + 7 = 3 \cdot 21 + 14 = 77.$$

If $n = 2^i \cdot 3^j \cdot 5^k$ for $i, j \geq 0$ and $k \geq 1$, then $\text{pow}(n) = 5^k$. Moreover, if $2^i \cdot 3^j \in \{1, 2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3\}$ and $1 \leq k \leq 3$, or if $2^i \cdot 3^j \in \{1, 2, 3, 2^2, 2 \cdot 3, 2^3\}$ and $k = 4$, or if $2^i \cdot 3^j = 1$ and $k = 5$; then $n = 2^i \cdot 3^j \cdot 5^k \leq 8 \cdot 5^4 = 5000 < 5300$. Thus

$$c \geq 8(1 + 2 + 3) + 6 \cdot 4 + 5 = 77.$$

Therefore $m = d = 77$.