Solutions
Solutions

Pascal Contest / Concours Pascal

PART A

1. $1-2+3-4+5-6 = (1+3+5)-(2+4+6)$
$= 9-12$
$= -3.$
ANSWER: (B)

2. $\sqrt{8\sqrt{4}} = \sqrt{8\times 2} = 4.$
ANSWER: (B)

3. Since 333 333 = 30(1111) + 33, 1111 is not an exact division of 333 333.
ANSWER: (E)

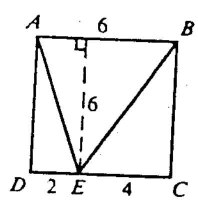
4. After four weeks, there will be $12+4=16$ boys and $8+2(4)=16$ girls in the club. Then there will be 32 club members.
ANSWER: (B)

5. Since the square of any odd integer is odd, 1989^2 is odd.
ANSWER: (D)

6. Since the average of the four numbers is 9, their sum is $4\times 9 = 36$. Then the fourth number is $36-(5+9+12)=10$. ANSWER: (D)

7. $\dfrac{3.9\times 10^5}{8.1\times 10^2}$ is approximately equal to $\dfrac{4\times 10^5}{8\times 10^2} = \dfrac{10^3}{2} = 500.$
ANSWER: (C)

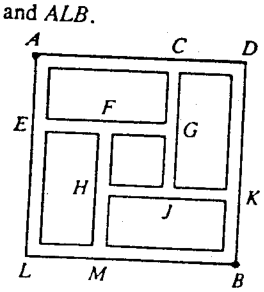
8. The area of $\triangle AEB = \frac{1}{2}(6)(6) = 18$



ANSWER: (C)

9. Since $\frac{1}{5} = \frac{8}{40}$ and $\frac{1}{4} = \frac{10}{40}$, then $\frac{n}{40}$ lies between $\frac{8}{40}$ and $\frac{10}{40}$.
A possible value of n is 9.
ANSWER: (B)

10. There are six routes he can follow, namely ADB, $ACJKB$, $AEGJKB$, $AEFHJKB$, $AEFMB$, and ALB.

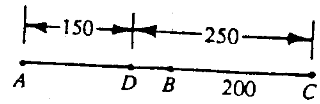


ANSWER: (B)

PART B

11. Since $\frac{5}{6}\times 72 = 60$, then $\frac{3}{4}$ of the original number is $\frac{3}{4}\times 72 = 54.$
ANSWER: (C)

12. Since $DC = 250$, then $DB = 250-200 = 50.$



ANSWER: (A)

13. $x^2 = x^4$ is satisfied by $x = 0$, 1, or -1.
ANSWER: (C)

14. Each number in the sequence leaves remainder 1 when divided by 3. Of the given answers only 9997 leaves remainder 1 when divided by 3.
ANSWER: (C)

15. The area of one of the circles is $\pi(1)^2 = \pi$. Then each of the regions A, B, C, D, E, F, and G has area $\frac{\pi}{4}$. The area of the pentagon is $D+E+F+G = 4\left(\frac{\pi}{4}\right) = \pi$.
ANSWER: (D)

16. Solution 1
The shirt numbers allowed are 3, 4, 5, 11, 12, 13, 14, 15, 21, 22, 23, 24, 25, 31, 32, 33, 34, 35, 41, 42, 43, 44, 45, 51, 52, 53, 54, and 55. Thus there are a total of 28 possible numbers.
Solution 2
There are only 3 possible shirt numbers with one digit. For shirts with two digits on them, there are 5 possibilities for the first digit and for each of these there are 5 possibilities for the second digit, giving $5\times 5 = 25$ possible 2-digit numbers. Thus there are $25+3=28$ possible shirt numbers.
ANSWER: (C)

17. Suppose Jane spends x dollars.
Then $4.50 - 2x = \frac{1}{2}(3-x)$
$9-4x = 3-x$
$3x = 6$
$x = 2.$
Thus Jane spent \$2 and Mary spent \$4.
ANSWER: (A)

18. If $3x+7 = 7x+15$
$4x = -8$
$x = -2.$
If $x^2 + k = 7x + 15$
$4+k = -14+15$
$k = -3.$
ANSWER: (B)

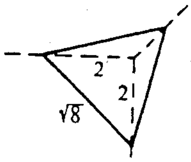
19. $3^{22} = 9^{11}$
$= 9(9^{10})$
$= 9(81)^5.$
Since 9 leaves remainder 4 on division by 5 and 81 leaves remainder 1 on division by 5, then 3^{22} leaves remainder $(4)(1^5) = 4$ on division by 5. Alternatively, we note that 81^5 ends in a 1, so $9(81)^5$ ends in a 9 and thus leaves remainder 4 on division by 5.
ANSWER (E)

20. In one second, at time 100 km/h, Nancy travels $\frac{100}{3600}$ km. In 10 seconds, she travels $10 \times \frac{1}{36} = \frac{5}{18}$ km. During this same time interval, Pat travels $\frac{1}{10} + \frac{5}{18} = \frac{17}{45}$ km. Ten seconds is equivalent to $\frac{10}{3600}$ hours. The speed of Pat's car is

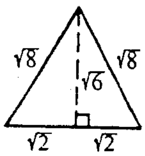
$\frac{17}{45} \div \frac{10}{3600} = 136$ km/h.

ANSWER: (E)

21. Each slice will create a new face which is an equilateral triangle with side of length $\sqrt{4+4} = \sqrt{8}$. Since the height of the triangle is $\sqrt{8-2} = \sqrt{6}$, its area is $\frac{1}{2}(\sqrt{8})(\sqrt{6}) = \frac{1}{2}\sqrt{48}$.



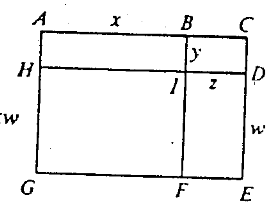
Each remaining face of the cuboctohedron is a square with side of length $\sqrt{8}$ and area $(\sqrt{8})^2 = 8$.
The total surface area of the six square faces and eight triangular faces is
$6(8) + 8\left(\frac{1}{2}\sqrt{48}\right) = 48 + 4\sqrt{48}$.



ANSWER: (A)

22. Let the lengths of AB, BI, ID, and DE be x, y, z, and w, respectively. The area of $ABIH = xy = 6$.
The area of $IDEF = zw = 15$.
The area of $ACEG = (AC)(CE)$

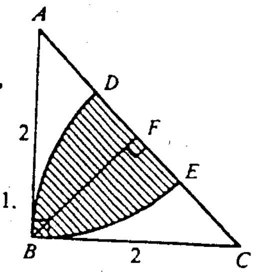
$= (x+z)(y+w)$
$= xy + zw + yz + xw$
$= 21 + yz + xw.$



We are given that x, y, z, and w are integers. Since $xy = 6$, the possible ordered pairs (x, y) are (3, 2), (2, 3), (6, 1), and (1, 6). Similarly $zw = 15$, so the possible ordered pairs (z, w) are (1, 15), (15, 1), (5, 3), and (3, 5). The maximum value of $yz + xw$ is obtained by selecting the pairs (1, 6) and (15, 1) or the pairs (6, 1) and (1, 15).
The maximum area for $ACEG$ is $21 + 90 + 1 = 112$.

ANSWER: (E)

23. Draw $BF \perp AC$. Since FBC is an isosceles right-angled triangle, $BF = FC = FA = \sqrt{2}$. Since $\angle C = 45°$, the area of sector $CBD = \frac{1}{8}\pi(2)^2 = \frac{\pi}{2}$.
The area of ΔFBC is $\frac{1}{2}(\sqrt{2})(\sqrt{2}) = 1$.
The area of shaded region FBD is $\frac{\pi}{2} - 1$.
Therefore, the shaded area is
$2\left(\frac{\pi}{2} - 1\right) = \pi - 2$.



ANSWER: (A)

24. Let $\sqrt{k-9} = x$ and $\sqrt{k+36} = y$.
Therefore, $k - 9 = x^2$ and $k + 36 = y^2$

$k = x^2 + 9$ and $k = y^2 - 36$
$x^2 + 9 = y^2 - 36$
$y^2 - x^2 = 45$
$(y+x)(y-x) = 45.$

We can tabulate possible values of $y + x$, $y - x$, y, x, and k.

$y+x$	$y-x$	y	x	k
45	1	23	22	493
15	3	9	6	45
9	5	7	2	13

There are three possible values of k.

ANSWER: (D)

25. Since a, b, and c are positive integers, the possible triples to consider are (1, 1, 12), (2, 2, 3), (1, 2, 6), and (1, 3, 4).
Because of the cyclic nature of $a^b + b^c + c^a$, the permutations of (1, 1, 12) and (2, 2, 3) will only produce one value each for the expression. These values are 14 and 21 respectively.
The permutations of (1, 2, 6) and (1, 3, 4) will each produce two values of $a^b + b^c + c^a$. For (1, 2, 6) these values are 39 and 71; for (1, 3, 4) they are 68 and 86. Therefore, there are only 6 possible values.

ANSWER: (B)

Cayley Contest / Concours Cayley

PART A

1. $4(5-2)^2 = 4(3^2) = 36.$

ANSWER: (B)

2. $15x + 20 = 25$

$15x = 5$
$x = \frac{1}{3}.$

ANSWER: (C)

3. See Pascal Contest, question 6.

ANSWER: (D)

4. After 4 days there are $16 + 4 = 20$ boys and $12 + 2(4) = 20$ girls in the class. Then there are a total of 40 students in the class.

ANSWER: (E)

5. Since the perimeter of the rectangle is $2y + 4$, the length plus the width is $y + 2$. Since the length is y, the width is 2 and the area is $2y$.

ANSWER: (A)

6. See Pascal Contest, question 11.

ANSWER: (C)

7. See Pascal Contest, question 10.

ANSWER: (B)

8. $a + c^{a-b} = 3 + (-2)^{3+3}$

$= 3 + 64$
$= 67.$

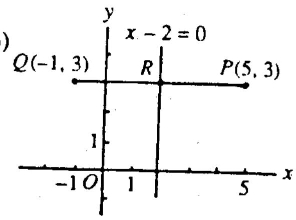
ANSWER: (A)

9. See Pascal Contest, question 16.

ANSWER: (B)

Solutions

10. Since $QR = RP$, the image of the point $P(5, 3)$ is the point $Q(-1, 3)$



ANSWER: (B)

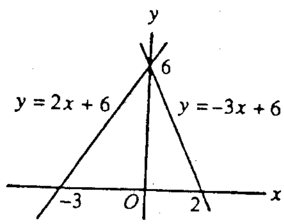
PART B

11. Let x represent the number of tapes that Mary has. Then Tim and Sue have $x - 5$ and $x - 12$ tapes, respectively.

$$x + (x - 5) + (x - 12) = 73$$
$$3x = 90$$
$$x = 30.$$

Hence, Mary has 30 tapes. ANSWER: (D)

12. The x-intercepts of the lines are -3 and 2. Therefore, the base of the triangle is 5 units long and its area is $\frac{1}{2}(5)(6) = 15$.



ANSWER: (C)

13. See Pascal Contest, question 15. ANSWER: (B)

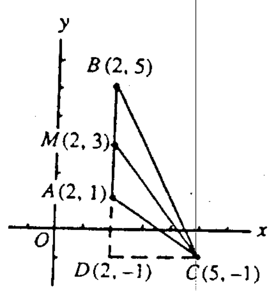
14. For the largest possible sum, the hundreds digits must be two 9s and one 8. Hence, the middle digit A is 1. If D and M are both 9s, DAM and MAD are both 919 and BAD is 819. This gives the maximum sum of $819 + 919 + 919 = 2657$. Note that if B is 9 and D and M are either 8 or 9, the sums are 2656 or 2655.

ANSWER: (D)

15. The midpoint of AB is $M(2, 3)$.

$$CM^2 = DC^2 + DM^2$$
$$= 3^2 + 4^2$$
$$= 25.$$

Therefore, $CM = 5$.



ANSWER: (A)

16.
$$a^2 + a = b^2 + b$$
$$a^2 - b^2 = b - a$$
$$(a + b)(a - b) = b - a.$$

Since $a \neq b$, $a + b = -1$. ANSWER: (C)

17. See Pascal Contest, question 20. ANSWER: (E)

18.
$$x + y - z + w$$
$$= (-a + b + c + d) + (a - b + c + d)$$
$$- (a + b - c + d) + (a + b + c - d)$$
$$= 4c.$$

Therefore, $4c = kc$ and $k = 4$ if $c \neq 0$. ANSWER: (E)

19. $m^n = 2^7 - 2^6 = 6^4$. The possible forms of m^n are 2^6, 4^3, 8^2, and 64^1. The possible values of x are 8, 7, 10, and 65 and the sum of these values is 90. ANSWER: (E)

20.
$$x = c + p,$$
$$y = b + r.$$
$$x + y = (b + p) + (c + r)$$
$$= -19 - 1$$
$$= -20.$$

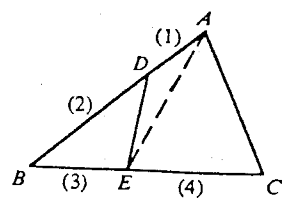
ANSWER: (B)

PART C

21. Join AE.

Area of $\Delta ADE = \frac{1}{2}$(area of ΔDBE)
$= 3.$

Area of $\Delta AEC = \frac{4}{3}$(area of ΔABE)
$= \frac{4}{3}(6 + 3)$
$= 12.$



Hence, the area of ΔABC is $12 + 9 = 21$. ANSWER: (D)

22. See Pascal Contest, question 23. ANSWER: (A)

23. See Pascal Contest, question 24. ANSWER: (D)

24. If $n = 1$, $\dfrac{6(1^5) + a(1^4) + b(1^3) - 1}{30} = 1^4$

$$6 + a + b - 1 = 30$$
$$a + b = 25. \quad (1)$$

If $n = 2$, $\dfrac{6(2^5) + a(2^4) + b(2^3) - 2}{30} = 1^4 + 2^4$

$$192 + 16a + 8b - 2 = 510$$
$$16a + 8b = 320$$
$$2a + b = 40. \quad (2)$$

Solving (1) and (2) gives $a = 15$, $b = 10$.

Therefore, $a - b = 5$. ANSWER: (D)

25. Suppose N has n digits and let X be the integer formed by deleting the units digit from N.

Thus, $N = 10X + 4$.

Let M be the integer formed by moving the units digit of N to the front of N.

Then, $M = 4(10^{n-1}) + X$.

But $M = 4N$.

Solutions

Therefore, $4(10^{n-1}) + X = 40X + 16$.

$$39X = 4(10^{n-1}) - 16$$

$$39X = 16[5^2(10^{n-3}) - 1].$$

For an integer value of X, 39 must divide $5^2(10^{n-3}) - 1$. If n is given values 3, 4, and 5, the corresponding values of $5^2(10^{n-3}) - 1$ are 24, 249, and 2499, none of which is divisible by 39. But if $n = 6$, $5^2(10^{n-3}) - 1 = 24999$
$= (39)(641)$.
Thus the smallest value of N has 6 digits.

Note that $N = 102564$ and $M = 410256$. One can actually create these numbers by noting that if N ends in 4, $M = 4N$ ends in 6. But the last digit is M is the second last digit in N. Therefore, the last two digits in N are 64. Then the last two digits of M are 56 and so the last three digits of N are 564. Continuing this process we can create the six digit integers N and M.

ANSWER: (C)

Fermat Contest / Concours Fermat

PART A

1. $-2 - (-2)^3 = -2 - (-8)$
$= 6.$

ANSWER: (A)

2. See Pascal Contest, question 6. ANSWER: (D)

3. $\sqrt{x} + \sqrt{16} = \sqrt{36}$
$\sqrt{x} + 4 = 6$
$\sqrt{x} = 2$
$x = 4.$

ANSWER: (B)

4. See Cayley Contest, question 5. ANSWER: (A)

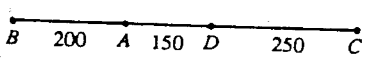
5. See Caylay Contest, question 10. ANSWER: (C)

6. See Pascal Contest, question 10. ANSWER: (B)

7. $x^2 = x^4$
$x^4 - x^2 = 0$
$x^2(x^2 - 1) = 0$
$x^2(x - 1)(x + 1) = 0$
$x = 0, 1, \text{or} -1.$

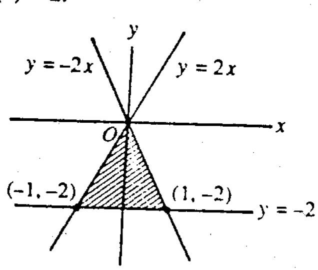
ANSWER: (B)

8. $BD = 200 + 150 = 350.$



ANSWER: (D)

9. The area of the triangle is $\frac{1}{2}(2)(2) = 2$.



ANSWER: (A)

10. Since the inequality must be satisfied by all values of x between 0 and 1, we can choose any particular value of x to test the answer. If $x = \frac{1}{2}$, $2x = 1$, $x^2 = \frac{1}{4}$, $2x^2 = \frac{1}{2}$, and $x^3 = \frac{1}{8}$.
The greatest of these is $2x$.

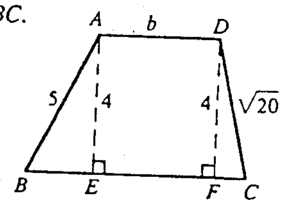
ANSWER: (B)

PART B

11. See Pascal Contest, question 16. ANSWER: (B)

12. See Pascal Contest, question 19. ANSWER: (A)

13. Draw AE and DF perpendicular to BC.
Since $AE = 4$, $BE = \sqrt{25 - 16} = 3$.
Since $DF = 4$, $FC = \sqrt{20 - 16} = 2$.
Therefore $b = EF = 9 - 3 - 2 = 4$.



ANSWER: (D)

14. See Pascal Contest, question 20. ANSWER: (E)

15. See Pascal Contest, question 15. ANSWER: (D)

16. $\frac{1}{a} + \frac{1}{b} = \frac{4}{a+b}$
$\frac{a+b}{ab} = \frac{4}{a+b}$
$a^2 + 2ab + b^2 = 4ab$
$a^2 - 2ab + b^2 = 0$
$(a - b)^2 = 0.$
Any ordered pair in which $a = b$, except (0,0), will satisfy the equation.

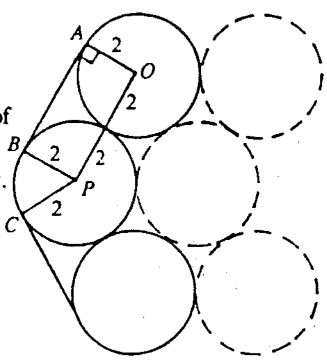
ANSWER: (E)

17. In ΔABC, $BC = 12$ and AD is the median to BC.
Therefore, $BD = DC = 6$.
Draw AE perpendicular to BC.
The area of
$\Delta ABC = \frac{1}{2}(12)(AE) = 24$
$AE = 4$.
Therefore, $ED = \sqrt{25-16} = 3$.
Hence, $BE = 3$ and $AB = \sqrt{16+9} = 5$
In ΔAEC, $AC = \sqrt{16+81} = \sqrt{97} > 5$.
Therefore, AB is the shortest side. ANSWER: (B)

18. See Pascal Contest, question 22. ANSWER: (E)

19. See Cayley Contest, question 19. ANSWER: (E)

20. In the diagram, $AB = OP = 4$.
Since there are 6 circles around the inner circle, arc BC is one-sixth of the circumference of one of the circles.
Therefore, arc $BC = \frac{1}{6}(4\pi) = \frac{2}{3}\pi$.
The total length of the band is
$6(AB + \text{arc } BC) = 6(4 + \frac{2}{3}\pi)$
$= 24 + 4\pi$.



ANSWER: (C)

PART C

21. The amount of light transmitted by each circle can be determined by finding the area of the unshaded portion of each circle.
Let the area of the shaded region be A. Then the larger circle transmits $(\pi a^2 - A)L$ lumens and the smaller circle transmits $(\pi b^2 - A)L$ lumens.
Since $(\pi a^2 - A)L - (\pi b^2 - A)L = \pi(a^2 - b^2)L$ the larger circle will transmit $\pi(a^2 - b^2)L$ more lumens than the smaller circle. ANSWER: (C)

22. See Cayley Contest, question 25. ANSWER: (C)

23. Let the sides of the tiles be x and y with $x > y$.
Then $nx^2 = (n + 76)\,y^2$
$x^2 = \left(\frac{n+76}{n}\right) y^2$
Since x, y, and n are integers, a solution can be obtained by finding perfect squares, p^2 and q^2, such that $p^2 - q^2 = 76$
$(p + q)(p - q) = 76$.

There are three possible systems of equations:

$p + q = 76$	$p + q = 19$	$p + q = 38$
$p - q = 1$	$p - q = 4$	$p - q = 2$.

The first two systems have no integral solutions. The third system has solution $p = 20$, $q = 18$ from which we obtain
$n = q^2 = 324$ and $n + 76 = p^2 = 400$. ANSWER: (D)

24. In base 9 the number is of the form $81a + 9b + c$.
In base 6 the number is of the form $36c + 6b + a$.
Equating these we get $81a + 9b + c = 36c + 6b + a$
$80a + 3b - 35c = 0$.
Since base 6 uses only the digits 0, 1, 2, 3, 4, and 5, we must find values of a, b, and c from these digits.
Since 5 divides $80a$ and $35c$, then 5 must divide $3b$.

Therefore, 5 divides b, so b can be either 0 or 5.
If $b = 0$, the equation becomes $80a - 35 = 0$
or $16a - 7c = 0$.
Neither a nor c is zero and none of the other values will satisfy the equation. Therefore, $b = 5$ and hence the middle digit is 5.
ANSWER: (E)

25. $|x| + |x - 2| + |y| = 6$.
There are 8 cases to consider of which only 6 give consistent results.

(i) $x \ge 2,\ y \ge 0$
$x + (x - 2) + y = 6$
$y = -2x + 8$.

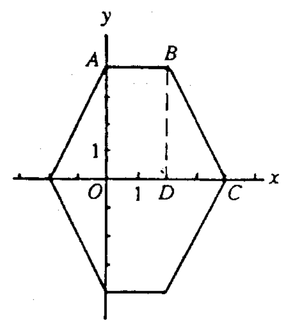
(ii) $x \ge 2,\ y < 0$
$x + (x - 2) - y = 6$
$y = 2x - 8$.

(iii) $0 \le x < 2,\ y \ge 0$
$x - (x - 2) + y = 6$
$y = 4$.

(iv) $0 \le x < 2,\ y < 0$
$x - (x - 2) - y = 6$
$y = -4$.

(v) $x < 0,\ y \ge 0$
$-x - (x - 2) + y = 6$
$y = 2x + 4$.

(vi) $x < 0,\ y < 0$
$-x - (x - 2) - y = 6$
$y = -2x - 4$.



The area of the region bounded by these six line segments is
2(rectangle $ABDO$) + 4(triangle BDC)
$= 2(8) + 4(4)$
$= 32$. ANSWER: (D)