

Optimization: Quadratic Maximum & Minimum Problems

A quadratic relation in standard form $y = ax^2 + bx + c$ can be rewritten in vertex form $y = a(x - p)^2 + q$ by creating a perfect square in the expression, then factoring the square. This technique is called **completing the square**.

- Completing the square allows you to find the maximum or minimum value of a quadratic relation algebraically, without using a graph which will be used in this lesson to determine the optimum value in a quadratic related situation.

Example 1:

Find the minimum product of two numbers whose difference is 8.

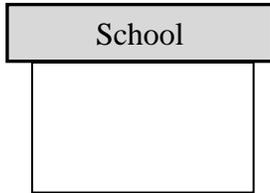
Example 2:

A glassworks company makes lead-crystal bowls, that creates a daily production cost C given by $C = 0.2b^2 - 10b + 650$, where b is the number of bowls made.

- a) How many bowls should be made to minimize the cost?
- b) What is the cost if this many bowls are made?

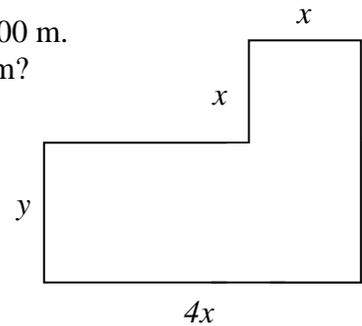
Example 3:

A rectangular fence is to be built around a playground, one side of the playground is against the school. If there is 400 m of fencing available, what dimensions would create the largest playground area?



Example 4:

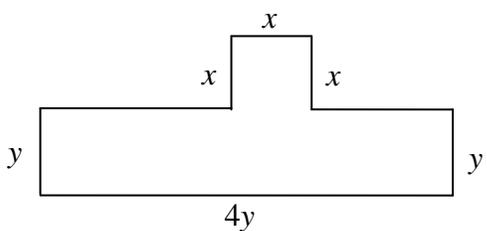
An enclosure is constructed with the shape shown and with a perimeter of 600 m. What are the values of x and y so that the area of the enclosure is a maximum?



Example 5:

Vehicles Incorporated currently sells an average of 20 compact cars each week at a price of \$6400 each. The sales department wants to increase the price, but the marketing department predicts that for every \$300 increase, sales will fall by one car. If the dealer cost (cost to the dealer) for each car is \$4000, what price will maximize profits for Vehicles Incorporated?

Exercise

- The general equation of a thrown object is given by $h = h_0 + v_0t - 5t^2$, where the values of h_0 and v_0 represent the initial height and initial speed of the object.
 - Determine the equation representing the height of a rock that is thrown upward from a cliff that is 15 m high, at an initial speed of 10 m/s.
 - Determine the maximum height of the rock.
- A ball is thrown from an apartment building. Its height, in metres, after t seconds, is given by $h = -5t^2 + 10t + 35$.
 - Determine the initial height of the ball.
 - Determine the maximum height of the ball.
 - Determine the length of time it takes for the ball to reach that height.
- Two numbers differ by 8. Their product is to have the least value possible. Determine the numbers.
- The sum of the base and the height of a triangle is 15 cm. What is the greatest possible area for a triangle having this property.
- A rectangular lot is bordered on one side by a stream and on the other three sides by fencing. If there is 600 metres of fence available, determine the dimensions of the lot with the greatest area.
- A rectangular field is enclosed by a fence and divided into two lots by another section of fence parallel to two of its sides. If the 600 metres of fence that is used must enclose a maximum area, what are the dimensions of the field?
- A fence is to be built around the area shown in the diagram. Determine the values of x and y that would produce a minimum area if the perimeter is 300 metres.
- If the total costs are $C(x) = 500 + 90x$, and total revenues are $R(x) = 150x - x^2$, where x represents the total number of merchandize sold.
 - Find the break-even point(s). (Revenue = Cost) or (Profit = 0)
 - Write the profit function, and find what level production maximizes the profit?
 - What is the maximum profit?
- Assume that a company knows that the cost to produce x items is given by the cost function $C(x) = 5x^2 + 800x$ dollars. It also knows that the revenue from x items is given by the revenue function $R(x) = 1000x + 200$. Find the maximum profit they can expect and how many of these items they have to produce and sell to make this maximum profit.

10. From producing a certain product, if total costs can be represented by $C(x) = 1600 + 1500x$, and the total revenues can be represented by $R(x) = 1600x - x^2$, find the break-even point(s) and the maximum possible profit.
11. An auditorium has seats for 1200 people. For the past several days, the auditorium has been filled to capacity for each show. Tickets currently cost \$5.00 and the owner wants to increase the ticket prices. He estimates that for each \$0.50 increase in price, 100 fewer people will attend. What ticket price will maximize the profit?
12. A grocer sells 50 loaves of bread a day. The cost is \$0.65 a loaf. The grocer estimates that for each \$0.05 price increase, 2 fewer loaves of bread will be sold. Graph, and then determine what cost will maximize the profit?
13. A bus company transports 500 people a day between Morse Rd. and high St. The one-way fare is \$0.50. The owner estimates that for each \$0.10 price increase, 50 passengers will be lost. Graph and then determine what price will maximize their profit?
14. A city transit system carries 24 800 bus riders per day for a fare of \$1.85. The city hopes to reduce car pollution by getting more people to ride the bus, while maximizing the transit system's revenue at the same time. A survey indicates that the number of riders will increase by 800 for every \$0.05 decrease in the fare. What fare will produce the greatest revenue?
15. A senior's dance club gas a \$5 cover charge and averages 300 customers on Friday nights. Over the past several months, the club has changed and cover price several times to see how it affects the number of customers. They have discovered that for every increase of \$0.50 in the cover charge, the number of customers decreases by 30, find the cover charge that maximize the revenue.
16. The cost C , in dollars, of operating a concrete-cutting machine is modelled by $C = 2.2n^2 - 66n + 655$, where n is the number of minutes the machine is run. How long must the machine run for the operating cost to be a minimum? What is the minimum cost?
17. A bus company has 4000 passengers daily, each paying a fare of \$2. For each \$0.15 increase, the company estimates that it will lose 40 passengers. If the company needs to take in \$10 450 per day to stay in business, what fare should be charged?

Answers

- 1) $h = -5t^2 + 10t + 15$; max height is 20m 2) 35m; 40m; 1s 3) 4, -4 4) $\frac{225}{8}$ or 28.125
- 5) 150 by 300 m; 45000 m² 6) 100 by 150 m 7) $\left(\frac{750}{29}, \frac{600}{29}\right)$
- 8a) (50, 5000), (10, 1400) b) 30 units c) Max Profit is \$400 9) \$2200 10) (20, 31600), (80, 121600)
- 11) \$5.50; Max income = \$6050 12) \$0.95; Max income = \$36.10 13) \$0.75; Max income = \$281.25
- 14) \$1.70 15) Keep the same price, no change 16) 15 min; Min cost is \$160
- 17) \$2.75 (3800 tickets) or \$14.25 (733.33 tickets)