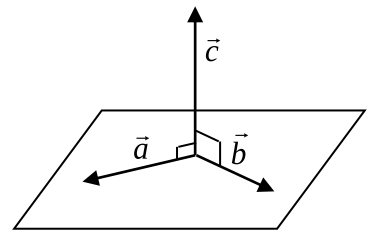
Cross Product of Two Vectors

What is Cross Product?



Cross Product is a **vector quantity** occurring in 3-space only.
It is a vector perpendicular to both vectors in the plane.

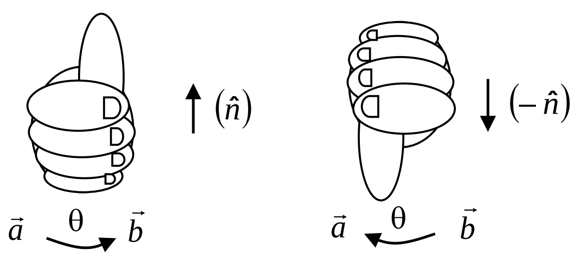
$\vec{c} \perp \vec{a}$ & $\vec{c} \perp \vec{b} \Leftrightarrow \vec{a} \times \vec{b}$ **(read as "$\vec{a}$ crosses $\vec{b}$")**

Cross Product in Geometric form

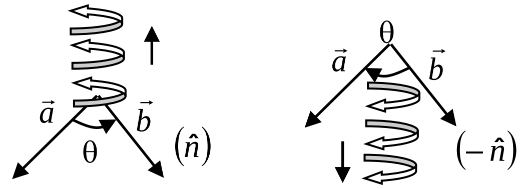
If $\vec{a}$ and $\vec{b}$ are two vectors with angle θ between them, then the <u>cross product</u> of $\vec{a}$ and $\vec{b}$, with respect to the diagram, is defined as:

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,(n)$; $\qquad \vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\sin\theta\,(-n)$

Right-hand system



Curl the direction of the angle up or down



Cross Product in Component form

If $\vec{a} = (x_1, y_1, z_1)$ $\quad \vec{b} = (x_2, y_2, z_2)$

$\vec{a} \times \vec{b} = (y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$

Properties of Cross Product

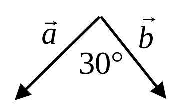
$\vec{p} \times \vec{q} = -(\vec{q} \times \vec{p})$

$\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$

$k(\vec{p} \times \vec{q}) = (k\vec{p}) \times \vec{q} = \vec{p} \times (k\vec{q})$

Example 1: Cross Product in Geometric form

In the figure, $|\vec{a}| = 8$, $|\vec{b}| = 6$, the angle between $\vec{a}$ and $\vec{b}$ is 30°, find



a) $\vec{a} \times \vec{b}$

$= |\vec{a}||\vec{b}|\sin\theta(\hat{n})$

$= (8)(6)\sin 30°(\hat{n})$

$= 24\hat{n}$

b) $\vec{b} \times \vec{a}$

$= |\vec{b}||\vec{a}|\sin\theta(-\hat{n})$

$= (6)(8)\sin 30°(-\hat{n})$

$= -24\hat{n}$

c) $|\vec{a} \times \vec{b}|$

$= |\vec{a}||\vec{b}|\sin\theta$

$= (8)(6)\sin 30°$

$= 24$

d) $|\vec{b} \times \vec{a}|$

$= |\vec{b}||\vec{a}|\sin\theta$

$= (6)(8)\sin 30°$

$= 24$

Example 2: Cross product in component form

If $\vec{a}$ = (4, 5, -3), $\vec{b}$ = (3, -7, 2), find

a) $\vec{a} \times \vec{b}$

$$\begin{array}{cccccccc} 4 & 5 & -3 & 4 & 5 & -3 \\ 3 & -7 & 2 & 3 & -7 & 2 \end{array}$$

$$\begin{array}{rrrr} & 10 & -9 & -28 \\ - & 21 & 8 & 15 \\ \hline & -11 & -17 & -43 \end{array}$$

b) $\vec{b} \times \vec{a}$

$$\begin{array}{cccccc} 3 & -7 & 2 & 3 & -7 & 2 \\ 4 & 5 & -3 & 4 & 5 & -3 \end{array}$$

$$\begin{array}{rrrr} & 21 & 8 & 15 \\ - & 10 & -9 & -28 \\ \hline & 11 & 17 & 43 \end{array}$$

$= -\left(\vec{a} \times \vec{b}\right)$

c) $\left|\vec{a} \times \vec{b}\right|$

$\sqrt{(-11)^2 + (-17)^2 + (-43)^2}$

$= \sqrt{2259}$

≈ 47.5

d) $\left|\vec{b} \times \vec{a}\right|$

$\sqrt{(11)^2 + (17)^2 + (43)^2}$

$= \sqrt{2259}$

≈ 47.5

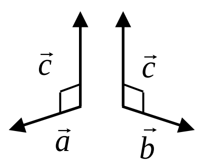
$= \left|\vec{a} \times \vec{b}\right|$

Example 3: Cross product applications

Find a unit vector $\vec{c}$ perpendicular to $\vec{a}$ = (4, -3, 1) and $\vec{b}$ = (2, 3, -1).

$\vec{c} \perp \vec{a}$ and $\vec{c} \perp \vec{b} \Leftrightarrow \vec{c} = \vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$ (2 possibilities)

$\vec{a} \times \vec{b} = (4,-3,1) \times (2,3,-1)$



$$\begin{array}{cccccc} 4 & -3 & 1 & 4 & -3 & 1 \\ 2 & 3 & -1 & 2 & 3 & -1 \end{array}$$

$$\begin{array}{rrrr} & 3 & 2 & 12 \\ - & 3 & -4 & -6 \\ \hline & 0 & 6 & 18 \end{array}$$

$\hat{c} = \frac{1}{|\vec{c}|}\vec{c}$

$= \frac{1}{|(0,6,18)|}(0,6,18)$

$= \frac{1}{\sqrt{0+36+324}}(0,6,18)$

$= \frac{1}{\sqrt{360}}(0,6,18)$

$= \frac{1}{6\sqrt{10}}(0,6,18)$

$= \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

$or \left(0, \frac{-1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)$

Example 4: Showing a property using Dot Product

Prove that $\left|\vec{a} \times \vec{b}\right| = \sqrt{(\vec{a} \bullet \vec{a})(\vec{b} \bullet \vec{b}) - (\vec{a} \bullet \vec{b})^2}$.

$\text{RS} = \sqrt{(\vec{a} \bullet \vec{a})(\vec{b} \bullet \vec{b}) - (\vec{a} \bullet \vec{b})^2}$

$= \sqrt{|\vec{a}|^2|\vec{b}|^2 - \left(|\vec{a}||\vec{b}|\cos\theta\right)^2}$

$= \sqrt{|\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta}$

$= \sqrt{|\vec{a}|^2|\vec{b}|^2\left(1-\cos^2\theta\right)}$

$= \sqrt{|\vec{a}|^2|\vec{b}|^2\left(\sin^2\theta\right)}$

$= |\vec{a}||\vec{b}|\sin\theta$

$= \left|\vec{a} \times \vec{b}\right| = \text{LS}$

LS = RS

Example 5: Cross product applications

Given $\vec{a}$ = (2, 1, 0), $\vec{b}$ = (-1, 0, 3) and $\vec{c}$ = (4, -1, 1), calculate the following triple scalar and triple vector products.

a) $\vec{a} \times \vec{b} \cdot \vec{c}$

$= (2,1,0) \times (-1,0,3) \cdot (4,-1,1)$

$\rightarrow$ Scalar

$$\begin{array}{cccccc} 2 & 1 & 0 & 2 & 1 & 0 \\ -1 & 0 & 3 & -1 & 0 & 3 \end{array}$$

$$\begin{array}{rrrr} & 3 & 0 & 0 \\ - & 0 & 6 & -1 \\ \hline & 3 & -6 & 1 \end{array}$$

$(3,-6,1) \cdot (4,-1,1)$

$= 12 + 6 + 1$

$= 19$

b) $\left(\vec{a} \cdot \vec{b}\right) \times \vec{c}$

$= (-2) \times \vec{c}$

DNE

Always do the cross product first

c) $(\vec{a} \times \vec{b}) \times \vec{c}$

$= (2,1,0) \times (-1,0,3) \times (4,-1,1)$

$\rightarrow$ Vector

Recall $\vec{a} \times \vec{b} = (3,-6,1)$

$$\begin{array}{cccccc} 3 & -6 & 1 & 3 & -6 & 1 \\ 4 & -1 & 1 & 4 & -1 & 1 \end{array}$$

$$\begin{array}{rrrr} & -6 & 4 & -3 \\ - & -1 & 3 & -24 \\ \hline & -5 & 1 & 21 \end{array}$$

d) $\vec{a} \times (\vec{b} \times \vec{c})$

$\vec{b} \times \vec{c} = (3,13,1)$

$\vec{a} \times (3,13,1)$

$(2,1,0) \times (3,13,1)$

$= (1,-2,23)$

$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Triple Product Expansion

Vector Triple Product: (Lagrange's Formula)

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}\left(\vec{a}\cdot\vec{b}\right)$$

"ABC = BAC – CAB"

Homework: P.407
#3,4abc,5,7,8b,11

Scalar Triple Product

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{b}\cdot(\vec{c}\times\vec{a}) \text{ or } -\vec{b}\cdot(\vec{a}\times\vec{c}) \text{ or } \vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$$

"ABC = – BAC or CAB"

Wedge Product: Exterior Product

In mathematics, the exterior **product** or **wedge product** of vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a}\wedge\vec{b}$ is an algebraic construction used in Euclidean geometry to study areas, volumes, and their higher-dimensional analogs