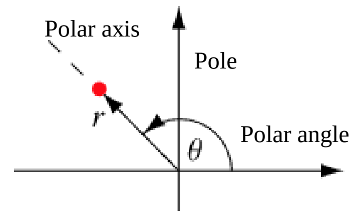
Polar Functions

Date:

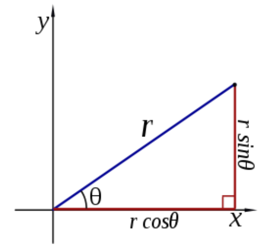
Polar Coordinate system

Polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a fixed point and an angle from a fixed direction.



The fixed point (analogous to the origin of a Cartesian system) is called the **pole**, and the ray from the pole with the fixed direction is the **polar axis**. The distance from the pole is called the **radial coordinate** or **radius**, and the angle is the **angular coordinate**, **polar angle**.

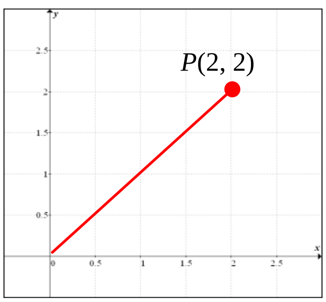
Converting between polar and Cartesian coordinates



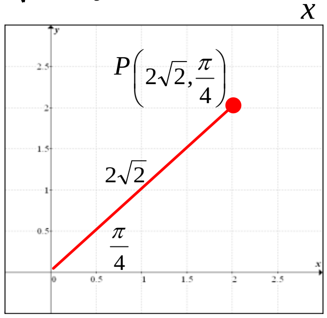
The two polar coordinates r and θ can be converted to the Cartesian coordinates x and y by using the trigonometric functions sine and cosine: $x = r\cos\theta \quad y = r\sin\theta$

while the two Cartesian coordinates x and y can be converted to the polar coordinates r and θ by $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$

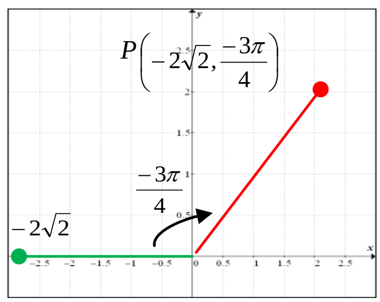
$$\theta = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \ge 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$



Rectangular coordinates



Polar coordinates (Direct Positive distance)

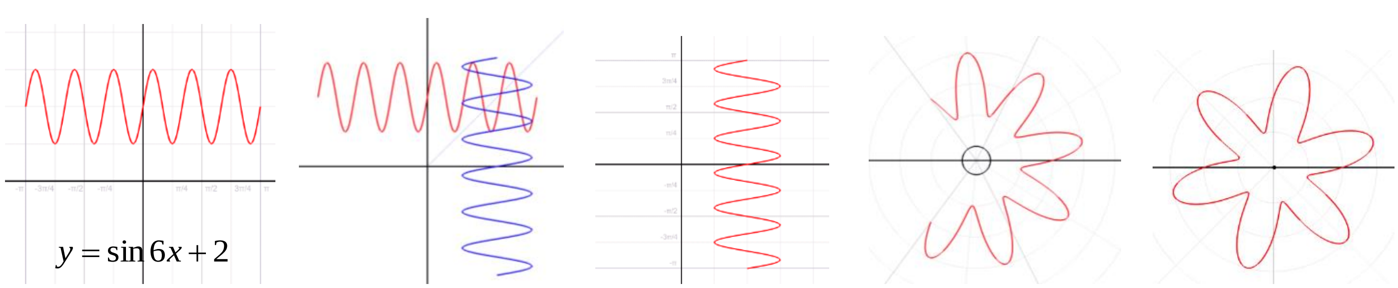


Polar coordinates (Direct Negative distance)

Note:

The directed negative distance $-2\sqrt{2}$ in the $\dfrac{-3\pi}{4}$ direction is the same as the direct positive distance $2\sqrt{2}$ in the $\dfrac{\pi}{4}$ direction. Thus the polar coordinates $\left(-2\sqrt{2}, \dfrac{-3\pi}{4}\right)$ and $\left(2\sqrt{2}, \dfrac{\pi}{4}\right)$ determine the same point.

Rectangular and Polar Coordinates



$r(\theta) = \sin 6\theta + 2$

http://upload.wikimedia.org/wikipedia/commons/archive/5/53/20100906050633%21Cartesian_to_polar.gif

Example 1: Rectangular and Polar Coordinates

$$(x,y)\to(r\cos\theta, r\sin\theta)$$

Find rectangular coordinates for the points with given polar coordinates.

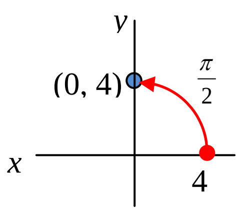
a) $\left(4,\frac{\pi}{2}\right)$

$x = r\cos\theta \qquad y = r\sin\theta$

$= 4\cos\frac{\pi}{2} \qquad = 4\sin\frac{\pi}{2}$

$= 0 \qquad = 4$

$\therefore \left(4,\frac{\pi}{2}\right)\to(0,4)$



b) $(-3,\pi)$

$x = r\cos\theta$

$= -3\cos\pi$

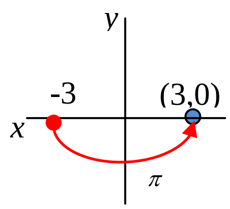
$= 3$

$y = r\sin\theta$

$= -3\sin\pi$

$= 0$

$\therefore (-3,\pi)\to(3,0)$



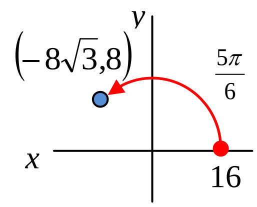
c) $\left(16,\frac{5\pi}{6}\right)$

$x = 16\cos\frac{5\pi}{6} \qquad y = 16\sin\frac{5\pi}{6}$

$= 16\left(\frac{-\sqrt{3}}{2}\right) \qquad = 16(0.5)$

$= -8\sqrt{3} \qquad = 8$

$\therefore \left(16,\frac{5\pi}{6}\right)\to\left(-8\sqrt{3},8\right)$



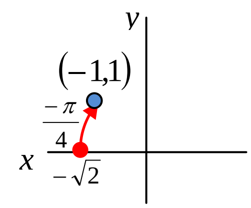
d) $\left(-\sqrt{2},\frac{-\pi}{4}\right)$

$x = -\sqrt{2}\cos\frac{-\pi}{4} \qquad y = -\sqrt{2}\sin\frac{-\pi}{4}$

$= -\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) \qquad = -\sqrt{2}\left(\frac{-\sqrt{2}}{2}\right)$

$= -1 \qquad = 1$

$\therefore \left(-\sqrt{2},\frac{-\pi}{4}\right)\to(-1,1)$



Example 2: Rectangular and Polar Coordinates

Recall: $(x,y)\to(r\cos\theta, r\sin\theta)$

Find two different sets of polar coordinates for the points with given rectangular coordinates.

a) $(1,0)$

$r = \sqrt{1^2+0^2} = 1$

$0 = 1\sin\theta$

$\sin\theta = 0$

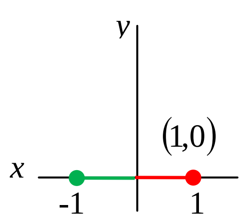
$\theta = 0 + 2k\pi, k\in I$

$or\ \theta = \pi + 2k\pi, k\in I$

$x = 1\cos\theta$

$x = 1$

(1,0) , $(1, 2\pi)$, $(-1, \pi)$



b) $(-3,3)$

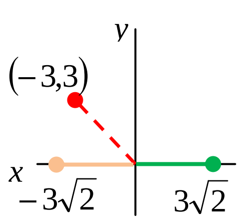
$r = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$\tan\theta = \frac{3}{-3} \quad \left(\frac{\pi}{4}\right)$

$\theta = \frac{3\pi}{4} + 2n\pi, n\in I$

$or\ \theta = \frac{-\pi}{4} + 2n\pi, n\in I$

$\left(3\sqrt{2},\frac{3\pi}{4}\right),\left(-3\sqrt{2},\frac{-\pi}{4}\right)$



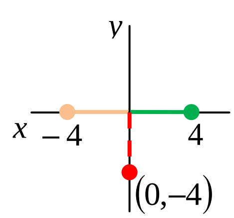
c) $(0,-4)$

$r = \sqrt{0+16} = 4$

$\tan\theta = \frac{-4}{0} \quad \left(\frac{-\pi}{2}\right)$

$\theta = \frac{-\pi}{2} + n\pi, n\in I$

$\left(4,\frac{-\pi}{2}\right),\left(4,\frac{3\pi}{2}\right),\left(-4,\frac{\pi}{2}\right)$



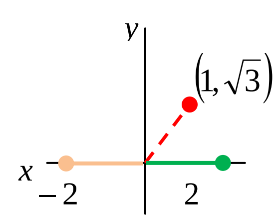
d) $\left(1,\sqrt{3}\right)$

$r = \sqrt{1+3} = 2$

$\tan\theta = \frac{\sqrt{3}}{1} \quad \left(\frac{\pi}{3}\right)$

$\theta = \frac{\pi}{3} + n\pi, n\in I$

$\left(2,\frac{\pi}{3}\right),\left(-2,\frac{4\pi}{3}\right),\left(-2,\frac{-2\pi}{3}\right)$



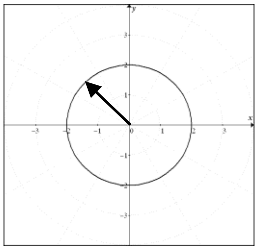
Example 3: Graphing with Polar Coordinates

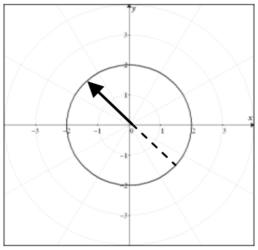
Graph all points in the plane that satisfy the given polar equation

a) $r = 2$ b) $r = -2$ c) $\theta = \dfrac{\pi}{6}$

a) The set of all points with directed distance 2 units from the pole is a circle of radius 2 centered at the origin.

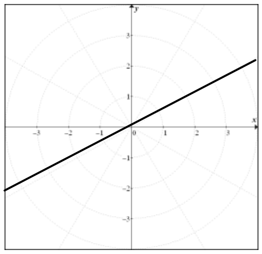
b) The set of all points with directed distance -2 units from the pole is a circle of radius 2 centered at the origin.





c) The set of all points of positive or negative directed distance from the pole in the $\dfrac{\pi}{6}$ direction is a line through the origin with slope $\tan\dfrac{\pi}{6} \rightarrow \dfrac{\sqrt{3}}{3}$

Line: $m = \dfrac{\sqrt{3}}{3}$ $y = \dfrac{\sqrt{3}}{3}x$



Example 4: Converting Polar to Rectangular

Use the polar-rectangular formulas to show that the polar graph of $r = 4\sin\theta$ is a circle.

$r = 4\sin\theta$

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

Multiply r to both sides

Recall : $y = r\sin\theta$

$x^2 + y^2 - 4y = 0$

$x^2 + y^2 - 4y + 4 = 4$

$x^2 + (y-2)^2 = 4$

The graph is a circle centered at $(0,2)$ with radius 2.

Method 2:

$r = 4\sin\theta$

$r^2 = 16\sin^2\theta$

$r^2 = 16\left(\dfrac{y^2}{r^2}\right)$

Square both sides

Recall : $\sin\theta = \dfrac{y}{r}$

$\sin^2\theta = \dfrac{y^2}{r^2}$

$x^2 + y^2 = r^2$

$x^2 + y^2 = \dfrac{16y^2}{r^2}$

$x^2 + y^2 = \dfrac{16y^2}{x^2 + y^2}$

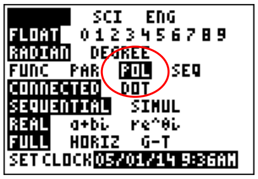
$(x^2 + y^2)^2 = 16y^2$

$x^2 + y^2 = 4y$

$x^2 + y^2 - 4y = 0$

$x^2 + y^2 - 4y + 4 = 4$

$x^2 + (y-2)^2 = 4$



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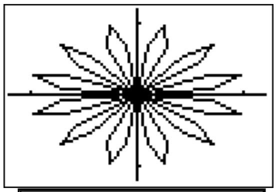
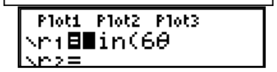


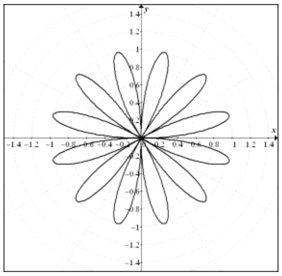


Parametric Equations of Polar Curves

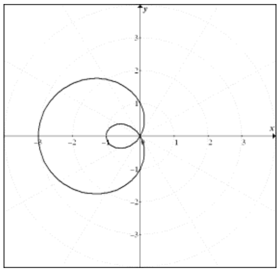
The polar graph $r = f(\theta)$ is the curve parametrically by:

$x = r\cos\theta = f(\theta)\cos\theta$ $y = r\sin\theta = f(\theta)\sin\theta$

Slopes of Polar Curves = $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$



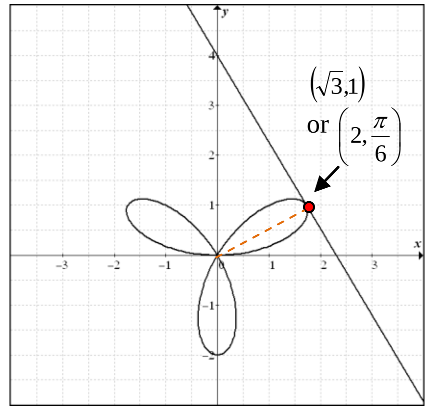
$r = \sin 6\theta$



$r = 1 - 2\cos\theta$

Example 5: Finding Slope of a Polar Curve

Find the slope of the rose curve $r = 2\sin 3\theta$ at the point where $\theta = \frac{\pi}{6}$ and use it to find the equation of the tangent line.



Recall:

$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$

$\because r = 2\sin 3\theta$

$\therefore x = 2\sin 3\theta\cos\theta \qquad y = 2\sin 3\theta\sin\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{6}} = \left.\frac{\frac{d}{d\theta}(2\sin 3\theta\sin\theta)}{\frac{d}{d\theta}(2\sin 3\theta\cos\theta)}\right|_{\theta=\frac{\pi}{6}}$$

$$= \frac{6\cos 3\theta\sin\theta + 2\sin 3\theta\cos\theta}{6\cos 3\theta\cos\theta + 2\sin 3\theta(-\sin\theta)}$$

$$= \frac{6\cos\frac{\pi}{2}\sin\frac{\pi}{6} + 2\sin\frac{\pi}{2}\cos\frac{\pi}{6}}{6\cos\frac{\pi}{2}\cos\frac{\pi}{6} - 2\sin\frac{\pi}{2}\sin\frac{\pi}{6}} = \frac{0 + 2(1)\left(\frac{\sqrt{3}}{2}\right)}{0 - 2(1)\left(\frac{1}{2}\right)} = -\sqrt{3}$$

Check : (Optional)

$x^2 = 4\sin^2 3\theta\cos^2\theta$

$y^2 = 4\sin^2 3\theta\sin^2\theta$

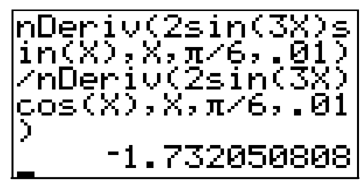
$x^2 + y^2 = 4\sin^2 3\theta\cos^2\theta + 4\sin^2 3\theta\sin^2\theta$

$r^2 = 4\sin^2 3\theta(\cos^2\theta + \sin^2\theta)$

$r^2 = 4\sin^2 3\theta(1) \quad (\sin^2 k\theta + \cos^2 k\theta = 1)$

$r = 2\sin 3\theta$

$\therefore k$ is not matter



$y = -\sqrt{3}x + b$

$1 = -\sqrt{3}(\sqrt{3}) + b$

$1 = -3 + b \qquad b = 4$

$\therefore$ The tangene line is $y = -\sqrt{3}x + 4$

or $y - 1 = -\sqrt{3}(x - \sqrt{3})$

When $\theta = \frac{\pi}{6}, x = 2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$

and $y = 2\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right) = 1$

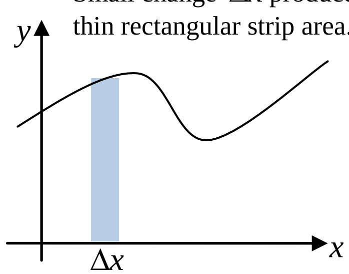
Point of tangency: $(\sqrt{3},1)$ or $\left(2,\frac{\pi}{6}\right)$ (polar)

Areas Enclosed by Polar Curves

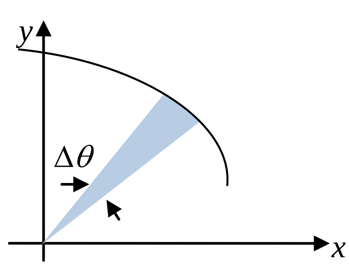
The area of the region between the origin and the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is

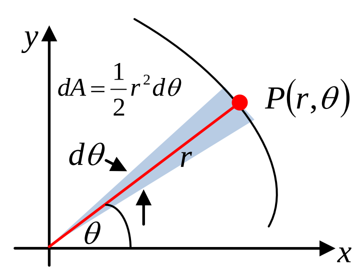
$$A = \int_\alpha^\beta \frac{1}{2}r^2\,d\theta = \int_\alpha^\beta \frac{1}{2}(f(\theta))^2\,d\theta$$

Small change Δx produces a thin rectangular strip area.



Small change $\Delta\theta$ produces a thin circular sector of area.





Recall: Area of sector of circle

$$A = \frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}\theta r^2$$

$$A(\theta) = \frac{r^2}{2}\theta$$

$$\frac{dA}{d\theta} = \frac{r^2}{2} \Rightarrow dA = \frac{r^2}{2}d\theta$$

Example 6: Area in Polar Coordinates

Find the area of the right semicircle with equation $r = 4\sin\theta$

The right semicircle is "swept out" as θ varies from 0 to $\frac{\pi}{2}$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (4\sin\theta)^2 d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (1-\cos 2\theta)d\theta$$

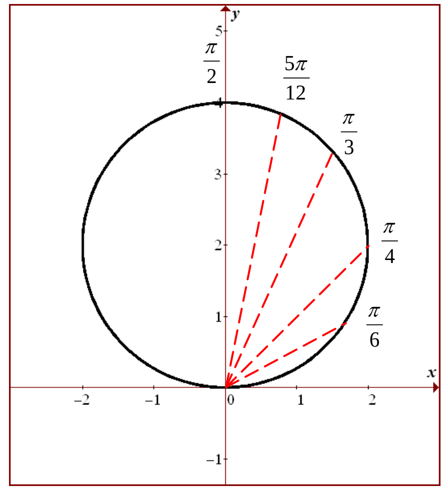
$$= (4\theta - 2\sin 2\theta)\Big]_0^{\frac{\pi}{2}}$$

$$= 4\left(\frac{\pi}{2}\right) - 0 = 2\pi$$

$$Semi-Circle\ Area = \frac{1}{2}\pi r^2$$

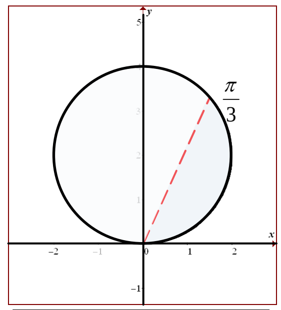
$$= \frac{1}{2}\pi(2)^2$$

$$= 2\pi$$

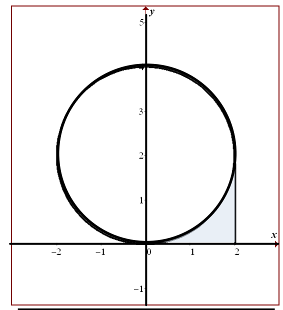


Keep in mind that the integral $\int_\alpha^\beta \frac{1}{2} r^2 d\theta$ does not compute the area *under* a curve, but rather the area "swept out" by the radial segment as θ varies from α to β.

For example, $\frac{1}{2}\int_0^{\frac{\pi}{3}} (4\sin\theta)^2 d\theta$ is equal to the area of the shaded region in the following figure A, not the area under the curve in figure B.



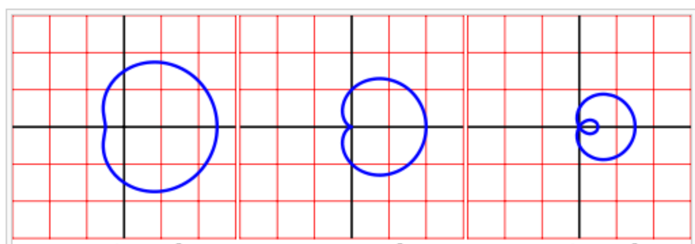
A: Region swept out by radial segment.



B: Region under the curve

Limaçon Curves

A **limaçon** or **limacon** is defined as a roulette formed when a circle rolls around the outside of a circle of equal radius. It can also be defined as the roulette formed when a circle rolls around a circle with half its radius so that the smaller circle is inside the larger circle.



$r = 1.5 + \cos\theta$ $r = 1 + \cos\theta$ $r = 0.5 + \cos\theta$

Example 7: Finding Area

Find the area of the region in the plane enclosed by the cardioids $r = 2(1+\cos\theta)$.

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2}[2(1+\cos\theta)]^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(4)(1+\cos\theta)^2 d\theta$$

$$= \int_0^{2\pi} 2(1+2\cos\theta+\cos^2\theta)d\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

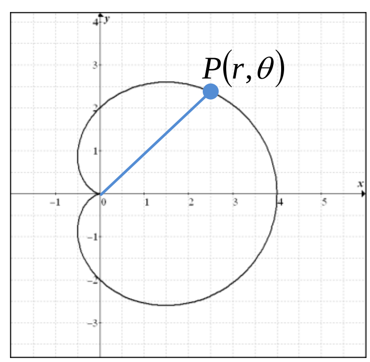
$$= \int_0^{2\pi} 2\left(1+2\cos\theta+\frac{\cos 2\theta+1}{2}\right)d\theta$$

$$= \int_0^{2\pi} (2+4\cos\theta+\cos 2\theta+1)d\theta$$

$$= \int_0^{2\pi} (3+4\cos\theta+\cos 2\theta)d\theta$$

$$= 3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\Bigg]_0^{2\pi}$$

$$= 6\pi - 0 = 6\pi \approx 18.8496$$

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fnInt(2(1+cos(X)
)²,X,0,2π)
         18.84955592
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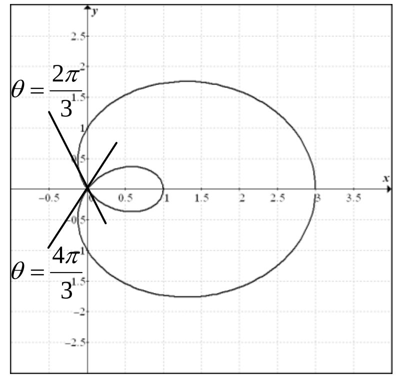
http://en.wikipedia.org/wiki/File:EpitrochoidIn1.gif

Example 8: Finding Area

Find the area inside the smaller loop of the limaçon $r = 2\cos\theta + 1$.



$r = 2\cos\theta + 1$

when polar axis $= 0$

$2\cos\theta + 1 = 0$

$\cos\theta = \frac{-1}{2} \rightarrow r = \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left[(2\cos\theta + 1)\right]^2 d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4\cos^2\theta + 4\cos\theta + 1\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left(4\left(\frac{\cos 2\theta + 1}{2}\right) + 4\cos\theta + 1\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left[2(\cos 2\theta + 1) + 4\cos\theta + 1\right] d\theta$$

$$= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2\cos 2\theta + 2 + 4\cos\theta + 1) d\theta$$

$$= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2\cos 2\theta + 4\cos\theta + 3) d\theta$$

$$= \frac{1}{2}\left(\sin 2\theta + 4\sin\theta + 3\theta\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\Bigg)$$

$$= \frac{1}{2}\left(\left[\sin 2\left(\frac{4\pi}{3}\right) + 4\sin\frac{4\pi}{3} + 3\left(\frac{4\pi}{3}\right)\right] - \left[\sin 2\left(\frac{2\pi}{3}\right) + 4\sin\frac{2\pi}{3} + 3\left(\frac{2\pi}{3}\right)\right]\right)$$

$$= \frac{1}{2}\left(2\sin\frac{4\pi}{3}\cos\frac{4\pi}{3} + 4\sin\frac{4\pi}{3} + 4\pi - 2\sin\frac{2\pi}{3}\cos\frac{2\pi}{3} - 4\sin\frac{2\pi}{3} - 2\pi\right)$$

$$= \frac{1}{2}\left[\left(\frac{-\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right) + 4\left(\frac{-\sqrt{3}}{2}\right) + 4\pi - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right) - 2\pi\right]$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 2\sqrt{3} + 4\pi + \frac{\sqrt{3}}{2} - 2\sqrt{3} - 2\pi\right)$$

$$= \frac{1}{2}\left(\sqrt{3} - 4\sqrt{3} + 2\pi\right) = \frac{1}{2}\left(-3\sqrt{3} + 2\pi\right) = \pi - \frac{3\sqrt{3}}{2} \approx 0.5435$$

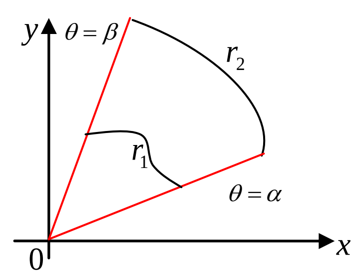
http://en.wikipedia.org/wiki/File:PedalCurve2.gif

```
0.5*fnInt((2cos(
X)+1)²,X,2π/3,4π
/3)
          .5435164422
```

Area Between Polar Curves

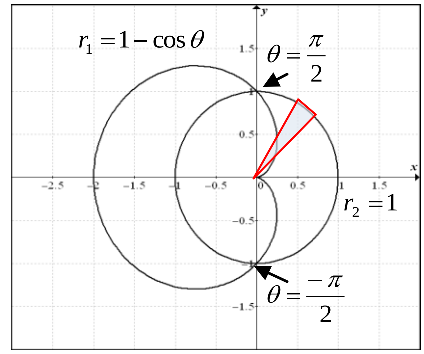
The area of the region between $r_1(\theta)$ and $r_2(\theta)$ for $\alpha \le \theta \le \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} {r_2}^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} {r_1}^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta$$



Example 9: Finding Area between Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.



Let $1 = 1 - \cos\theta$

$\cos\theta = 0$

$\theta = \frac{\pi}{2} \ \& \ \frac{3\pi}{2} \ or \ \frac{-\pi}{2}$

$$A = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right)d\theta = 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(r_2^{\,2} - r_1^{\,2}\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \left(1 - 2\cos\theta + \cos^2\theta\right)\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(2\cos\theta - \cos^2\theta\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(2\cos\theta - \frac{\cos 2\theta + 1}{2}\right)d\theta$$

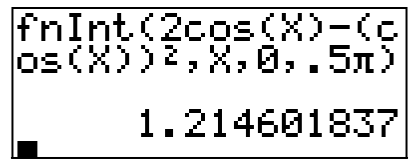
$$= \int_0^{\frac{\pi}{2}} \left(2\cos\theta - \frac{1}{2}\cos 2\theta - \frac{1}{2}\right)d\theta$$

$$= 2\sin\theta - \frac{1}{4}\sin 2\theta - \frac{1}{2}\theta\Big]_0^{\frac{\pi}{2}}$$

$$= \left[2\sin\frac{\pi}{2} - \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{2}\right)\right] - \left[2\sin 0 - \frac{1}{4}\sin 0 - \frac{1}{2}(0)\right]$$

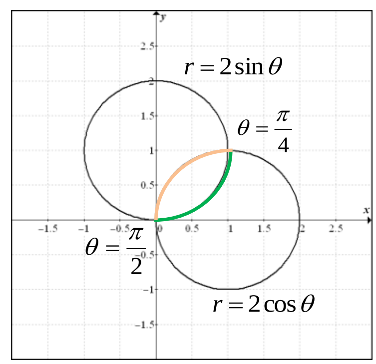
$$= 2(1) - 0 - \frac{\pi}{4}$$

$$= 2 - \frac{\pi}{4} \approx 1.215$$



Example 10: Area bounded by Curves

Find the total area bounded by the curves $r = 2\cos\theta$ and $r = 2\sin\theta$



Region A: Area swept out by

$r = 2\sin\theta$ for $0 \le \theta \le \frac{\pi}{4}$

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}\left[2\sin\theta\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(4\sin^2\theta\right)d\theta = \int_0^{\frac{\pi}{4}} 2\left(\sin^2\theta\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\left(\frac{1 - \cos 2\theta}{2}\right)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta\right)d\theta$$

$$= \theta - \frac{1}{2}\sin 2\theta\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

Region B: Area swept out by

$r = 2\cos\theta$ for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\left[2\cos\theta\right]^2 d\theta$$

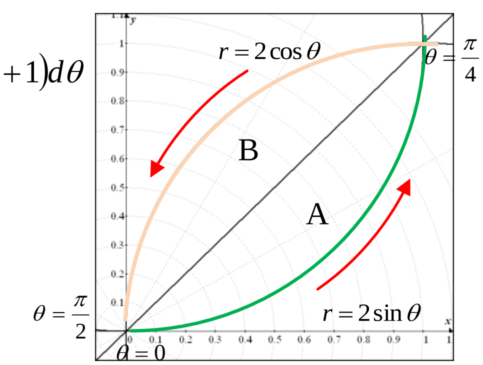
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\left(4\cos^2\theta\right)d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\left(\cos^2\theta\right)d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\left(\frac{\cos 2\theta + 1}{2}\right)d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cos 2\theta + 1\right)d\theta$$

$$= \frac{1}{2}\sin 2\theta + \theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$



Total area = $\left(\frac{\pi}{4} - \frac{1}{2}\right) + \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$

Arc length for Polar Curves

For a polar graph defined on a interval (α, β), if the graph does not retrace itself in that interval and if $\frac{dr}{d\theta}$ is continuous, then the length of the arc from $\theta = \alpha$ to $\theta = \beta$ is $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Proof:

Recall: The arc length of a curve defined parametrically by $x = x(t)$ and $y = y(t)$, for $a \le t \le b$ is given by:

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ **(Section 10.2)**

Once again thinking of a polar curve as a parametric representation (where the parameter is θ), we have that for the polar curve $r = f(\theta)$, recall: $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$

This gives us:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2$$

$$= \left\{[f'(\theta)]^2\cos^2\theta - 2f'(\theta)f(\theta)\sin\theta\cos\theta + [f(\theta)]^2\sin^2\theta\right\} + \left\{[f'(\theta)]^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + [f(\theta)]^2\cos^2\theta\right\}$$

$$= [f'(\theta)]^2\cos^2\theta + [f'(\theta)]^2\sin^2\theta + [f(\theta)]^2\sin^2\theta + [f(\theta)]^2\cos^2\theta$$

$$= [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) + [f(\theta)]^2(\sin^2\theta + \cos^2\theta)$$

$$= [f'(\theta)]^2 + [f(\theta)]^2 = [f'(\theta)]^2 + r^2$$

Example 11: Finding Arc length for Polar Curves

Find the length of the spiral $r = e^{\theta}$ from $\theta = 0$ to $\theta = \pi$.

$r = e^{\theta}$

$\frac{dr}{d\theta} = e^{\theta}$

$\left(\frac{dr}{d\theta}\right)^2 = e^{2\theta}$

$r^2 = (e^{\theta})^2 = e^{2\theta}$

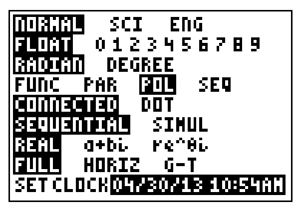
$$L = \int_0^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + (r)^2}\, d\theta$$

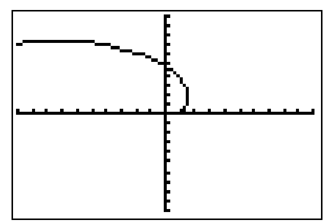
$$L = \int_0^{\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta$$

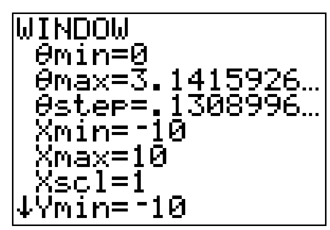
$$= \int_0^{\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

$$= \int_0^{\pi} \sqrt{2e^{2\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi} e^{\theta}\, d\theta = \sqrt{2}e^{\theta}\Big]_0^{\pi} = \sqrt{2}e^{\pi} - \sqrt{2}$$







Example 12: Finding Arc length for Polar Curves

Find the length of the spiral $r = 2 - 2\cos\theta$.

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = 4\sin^2\theta$$

$$r^2 = (2 - 2\cos\theta)^2$$

$$= 4 - 8\cos\theta + 4\cos^2\theta$$

Notice the curve rraced out with $0 \le \theta \le 2\pi$

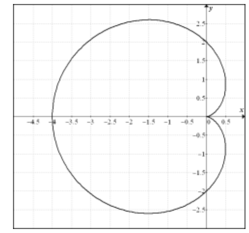
$$L = \int_0^{2\pi} \sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4 - 8\cos\theta + 4} = \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8(1 - \cos\theta)}\, d\theta = \int_0^{2\pi} \sqrt{8\left(2\sin^2\frac{\theta}{2}\right)}\, d\theta = \int_0^{2\pi} 4\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= 4\left|-2\cos\frac{\theta}{2}\right|_0^{2\pi} = 8\left|\cos\frac{\theta}{2}\right|\Bigg]_0^{2\pi} = 8|\cos\pi - \cos 0| = 8|-1-1|$$

$$= 8|-2| = 8(2) = 16$$



$$\because \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\therefore \sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$$

$$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$$

Recall:
$\sqrt{x^2} = |x|$

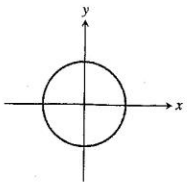
Polar Functions

Date:

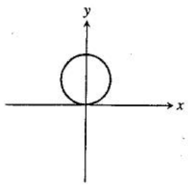
A SMALL POLAR GALLERY

Here are a few of the more common polar graphs and the θ-intervals that can be used to produce them.

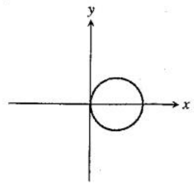
CIRCLES



$r = \text{constant}$
$0 \le \theta \le 2\pi$

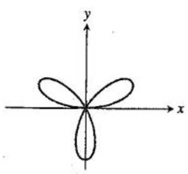


$r = a \sin \theta$
$0 \le \theta \le \pi$

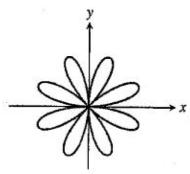


$r = a \cos \theta$
$0 \le \theta \le \pi$

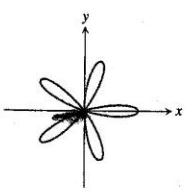
ROSE CURVES



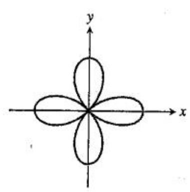
$r = a \sin n\theta$, n odd
$0 \le \theta \le \pi$
n petals
y-axis symmetry



$r = a \sin n\theta$, n even
$0 \le \theta \le 2\pi$
$2n$ petals
y-axis symmetry and x-axis symmetry



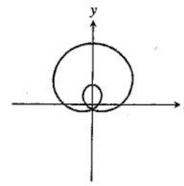
$r = a \cos n\theta$, n odd
$0 \le \theta \le \pi$
n petals
x-axis symmetry



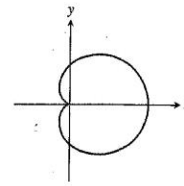
$r = a \cos n\theta$, n even
$0 \le \theta \le 2\pi$
$2n$ petals
y-axis symmetry and x-axis symmetry

LIMAÇON CURVES

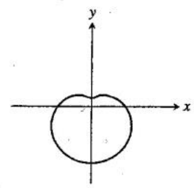
$r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ with $a > 0$ and $b > 0$
($r = a \pm b \sin \theta$ has y-axis symmetry; $r = a \pm b \cos \theta$ has x-axis symmetry.)



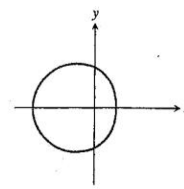
$\frac{a}{b} < 1$
$0 \le \theta \le 2\pi$
Limaçon with loop



$\frac{a}{b} = 1$
$0 \le \theta \le 2\pi$
Cardioid

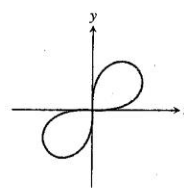


$1 < \frac{a}{b} < 2$
$0 \le \theta \le \pi$
Dimpled limaçon

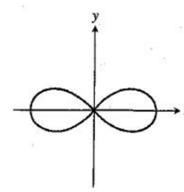


$\frac{a}{b} \ge 2$
$0 \le \theta \le 2\pi$
Convex limaçon

LEMNISCATE CURVES

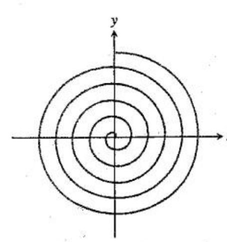


$r^2 = a^2 \sin 2\theta$
$0 \le \theta \le \pi$



$r^2 = a^2 \cos 2\theta$
$0 \le \theta \le \pi$

SPIRAL OF ARCHIMEDES



$r = \theta > 0$

Homework:
P. 557 #1 - 57