Derivatives of Inverse Trigonometric Functions

Date:

Derivative of the Inverse Trigonometric Functions (Arc-Trigonometric)

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}x=\frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$$

Proof: Arc-sine

$y=\sin^{-1}x$

$\Rightarrow x=\sin y; \frac{-\pi}{2}\le y\le\frac{\pi}{2}$

$\frac{d}{dx}(x)=\frac{d}{dx}\sin y$

$1=\cos y\frac{dy}{dx}$

If $y\ne\pm\frac{\pi}{2}, \frac{dy}{dx}=\frac{1}{\cos y}$

$\frac{dy}{dx}=\frac{1}{\sqrt{1-\sin^2 y}}$

$\because x=\sin y \quad \therefore \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

Proof: Arc-cosine

$y=\cos^{-1}x$

$\Rightarrow x=\cos y, 0\le y\le\pi$

$\frac{d}{dx}(x)=\frac{d}{dx}\cos y$

$1=-\sin y\frac{dy}{dx}$

If $y\ne 0$ or $\pi, \frac{dy}{dx}=\frac{-1}{\sin y}$

$\frac{dy}{dx}=\frac{-1}{\sqrt{1-\cos^2 y}}$

$\because x=\cos y \quad \therefore \frac{dy}{dx}=\frac{-1}{\sqrt{1-x^2}}$

Proof: Arc-tangent

$y=\tan^{-1}x$

$\Rightarrow x=\tan y; \frac{-\pi}{2}<y<\frac{\pi}{2}$

$\frac{d}{dx}(x)=\frac{d}{dx}\tan y$

$1=\sec^2 y\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{1}{\sec^2 y}$

$\frac{dy}{dx}=\frac{1}{1+\tan^2 y}$

$\because x=\tan y \quad \therefore \frac{dy}{dx}=\frac{1}{1+x^2}$

Recall

$\sin^2 y+\cos^2 y=1$

$\cos y=+\sqrt{1-\sin^2 y}$

$\frac{-\pi}{2}<y<\frac{\pi}{2}$

Recall

$\sin^2 y+\cos^2 y=1$

$\sin y=+\sqrt{1-\cos^2 y}$

$0<y<\pi$

Recall

$\sec^2 y=1+\tan^2 y$

$\tan^2 y=\sec^2 y-1$

Recall

$\csc^2 y=1+\cot^2 y$

$\cot^2 y=\csc^2 y-1$

Example 1: Derivatives of Inverse Trigonometric functions

Find the derivative of the following

a) $y=\sin^{-1}(1-x^2)$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(1-x^2)^2}}\frac{d}{dx}(1-x^2)$$

$$=\frac{-2x}{\sqrt{1-(1-2x^2+x^4)}}=\frac{-2x}{\sqrt{2x^2-x^4}}$$

$$=\frac{-2x}{\sqrt{x^2(2-x^2)}}=\frac{-2x}{|x|\sqrt{2-x^2}}$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{2-x^2}} \text{ if } x<0$$

$$\frac{dy}{dx}=\frac{-2}{\sqrt{2-x^2}} \text{ if } x>0$$

b) $y=x\tan^{-1}\sqrt{x}$

$$\frac{dy}{dx}=(1)\left(\tan^{-1}\sqrt{x}\right)+(x)\frac{1}{1+\left(\sqrt{x}\right)^2}\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=\tan^{-1}\sqrt{x}+\left(\frac{x}{1+x}\right)\left(\frac{1}{2}x^{\frac{-1}{2}}\right)$$

$$=\tan^{-1}\sqrt{x}+\frac{\sqrt{x}}{2(1+x)}$$

Domain: *given* $\sqrt{x}, x>0$

Only 1 case

$\frac{x}{1+|x|}\rightarrow\frac{x}{1+x}$

Example 2: Derivatives of combined Inverse-Trig Functions

Differentiate $y=\cos^{-1}(\sin x)$ and use the result to sketch the graph of the derivatives and then the function.

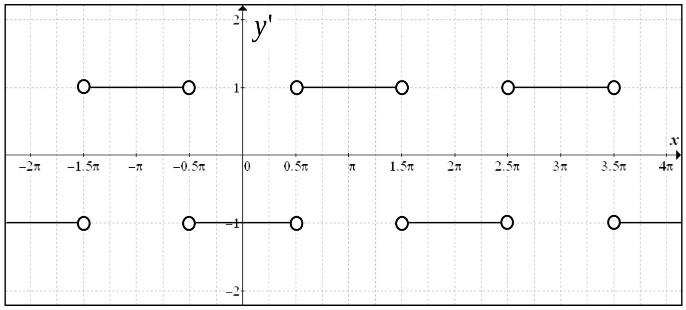
$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-\sin^2 x}}\frac{d}{dx}(\sin x)$$

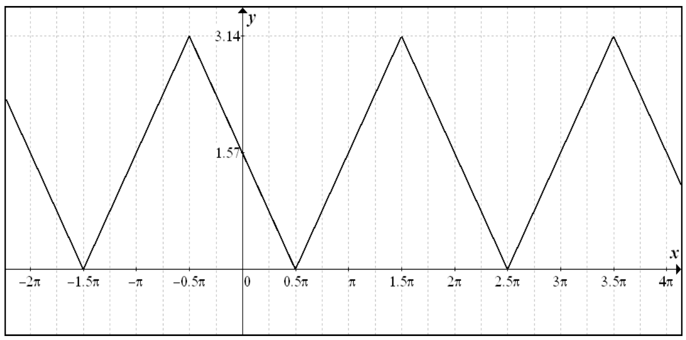
$$=\frac{-1}{\sqrt{\cos^2 x}}(\cos x)$$

$$=\frac{-\cos x}{|\cos x|},\cos x\neq 0$$

$$\frac{dy}{dx}=1 \text{ if } \cos x<0$$

$$\frac{dy}{dx}=-1 \text{ if } \cos x>0$$





$$\because \frac{dy}{dx}=\pm 1 \quad \therefore y=\pm x+c$$

To find c, find the y - int

$$y=\cos^{-1}(\sin x)$$

$$y-\text{int}: x=0$$

$$y=\cos^{-1}(\sin 0)$$

$$y=\cos^{-1}0$$

$$\cos y=0, 0\leq y\leq \pi$$

$$y=\frac{\pi}{2}\rightarrow c=\frac{\pi}{2}$$

$$\therefore y=\pm x+\frac{\pi}{2}$$

Example 3: Derivatives of Inverse Trigonometric functions

Find the derivative of $y=\tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{dy}{dx}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx}=\frac{1}{1+\frac{x^2}{y^2}}\frac{1y-x\frac{dy}{dx}}{y^2}$$

$$\frac{dy}{dx}=\frac{y^2}{x^2+y^2}\frac{y-x\frac{dy}{dx}}{y^2}$$

$$\frac{dy}{dx}=\frac{y-x\frac{dy}{dx}}{x^2+y^2}$$

$$\left(x^2+y^2\right)\frac{dy}{dx}=y-x\frac{dy}{dx}$$

$$\left(x^2+y^2+x\right)\frac{dy}{dx}=y$$

$$\frac{dy}{dx}=\frac{y}{x^2+y^2+x}$$

Example 4: Applications of Inverse Trigonometric Functions

A particle moves along the x-axis so that its position at any time $t\geq 0$ is $x(t)=\tan^{-1}\sqrt{t}$. What is the velocity of the particle when t =16? Distance is in *m*, time in *sec*.

Recall : $v(t)=x'(t)$

$$v(t)=\frac{dx}{dt}=\frac{1}{1+\sqrt{t}^2}\frac{d}{dt}\left(\sqrt{t}\right)$$

$$=\frac{1}{1+t}\frac{1}{2\sqrt{t}}$$

when $t=16$

$$v(16)=\frac{1}{1+16}\cdot\frac{1}{2\sqrt{16}}$$

$$=\frac{1}{17}\cdot\frac{1}{8}=\frac{1}{136}$$

$\therefore$ Velocity of the particle at t = 16 is $\frac{1}{136}$ m/sec.

Derivatives of the Other Three Inverse Trig Functions

$$\frac{d}{dx}\sec^{-1}x=\begin{cases}\dfrac{1}{x\sqrt{x^2-1}}, x>1\\ \dfrac{-1}{x\sqrt{x^2-1}}, x<-1\end{cases}\qquad \frac{d}{dx}\csc^{-1}x=\begin{cases}\dfrac{-1}{x\sqrt{x^2-1}}, x>1\\ \dfrac{1}{x\sqrt{x^2-1}}, x<-1\end{cases}\qquad \frac{d}{dx}\cot^{-1}x=\frac{-1}{1+x^2}$$

Proof: Arc-secant

$y=\sec^{-1}x\Rightarrow x=\sec y;$

$0\le y\le\frac{\pi}{2}\ \&\ \frac{\pi}{2}\le y\le\pi$

$\frac{d}{dx}(x)=\frac{d}{dx}\sec y$

$1=\sec y\tan y\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{1}{\sec y\tan y}$

$\because x=\sec y$

and $\tan y=\pm\sqrt{\sec^2 y-1}$

$\tan y=\pm\sqrt{x^2-1}$

$\therefore\frac{dy}{dx}=\frac{1}{\pm x\sqrt{x^2-1}}$

$\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$ if $x>1$

$\frac{d}{dx}\sec^{-1}x=\frac{-1}{x\sqrt{x^2-1}}$ if $x<-1$

Proof: Arc-cosecant

$y=\csc^{-1}x\Rightarrow x=\csc y;$

$0\le y\le\frac{\pi}{2}\ \&\ \frac{-\pi}{2}\le y\le 0$

$\frac{d}{dx}(x)=\frac{d}{dx}\csc y$

$1=-\csc y\cot y\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{-1}{\csc y\cot y}$

$\because x=\csc y$

and $\cot y=\pm\sqrt{\csc^2 y-1}$

$\cot y=\pm\sqrt{x^2-1}$

$\therefore\frac{dy}{dx}=\frac{-1}{\pm x\sqrt{x^2-1}}$

$\frac{d}{dx}\csc^{-1}x=\frac{-1}{x\sqrt{x^2-1}}$ if $x>1$

$\frac{d}{dx}\csc^{-1}x=\frac{1}{x\sqrt{x^2-1}}$ if $x<-1$

Proof: Arc-cotangent

$y=\cot^{-1}x\Rightarrow x=\cot y;$

$0\le y\le\pi$

$\frac{d}{dx}(x)=\frac{d}{dx}\cot y$

$1=-\csc^2 y\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{-1}{\csc^2 y}$

$\because\csc^2 y=1+\cot^2 y$

$\therefore\frac{dy}{dx}=\frac{-1}{1+\cot^2 y}$

$\because x=\cot y$

$\therefore\frac{dy}{dx}=\frac{-1}{1+x^2}$

Recall
$\sec^2 y=1+\tan^2 y$
$\tan^2 y=\sec^2 y-1$

Recall
$\csc^2 y=1+\cot^2 y$
$\cot^2 y=\csc^2 y-1$

Example 5: Derivative of the Co-Inverse Trig Functions

Find the derivative of $\sec^{-1}(5x^4)$

$$\frac{d}{dx}\sec^{-1}(5x^4)=\frac{1}{|5x^4|\sqrt{(5x^4)^2-1}}\frac{d}{dx}(5x^4)$$

Always positive

$$=\frac{20x^3}{5x^4\sqrt{25x^8-1}}=\frac{4}{x\sqrt{25x^8-1}}$$

Derivatives of Inverse Trigonometric Functions Date:

Inverse Function-Inverse Cofunction Identities (Proofs of the identities can be determined graphically)

$\cos^{-1}x = \dfrac{\pi}{2} - \sin^{-1}x$ $\qquad \cot^{-1}x = \dfrac{\pi}{2} - \tan^{-1}x$ $\qquad \csc^{-1}x = \dfrac{\pi}{2} - \sec^{-1}x$

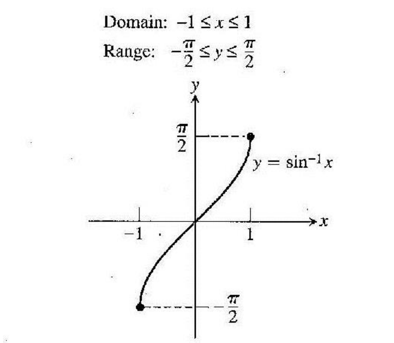
$\sec^{-1}x = \cos^{-1}\left(\dfrac{1}{x}\right)$

$\csc^{-1}x = \sin^{-1}\left(\dfrac{1}{x}\right)$

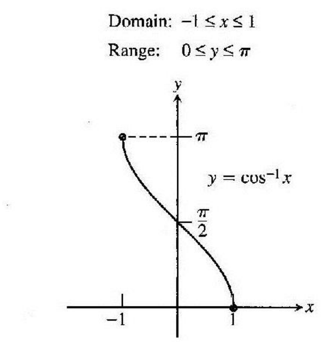
$$\boxed{\begin{aligned}&\text{Proof : Let } \sec^{-1}x = ?\\ &x = \sec ? \quad 0 \le y < \frac{\pi}{2}, \frac{\pi}{2} < y \le \pi\\ &x = \frac{1}{\cos ?}\\ &\cos ? = \frac{1}{x} \Rightarrow ? = \cos^{-1}\left(\frac{1}{x}\right)\end{aligned}}$$

$$\boxed{\begin{aligned}&\text{Proof : Let } \csc^{-1}x = ?\\ &x = \csc ? \quad \frac{-\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0\\ &x = \frac{1}{\sin ?}\\ &\sin ? = \frac{1}{x} \Rightarrow ? = \sin^{-1}\left(\frac{1}{x}\right)\end{aligned}}$$

Recall:



(a)



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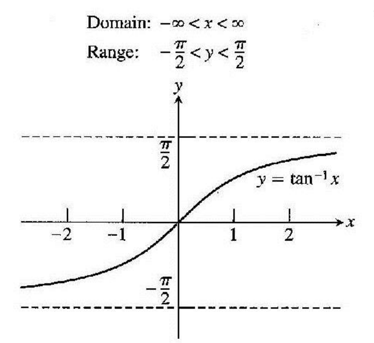

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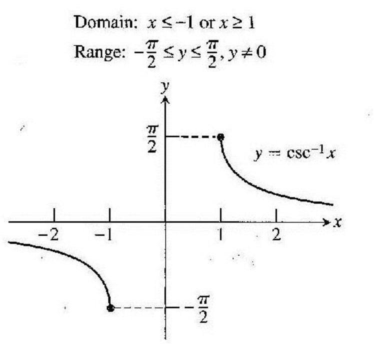

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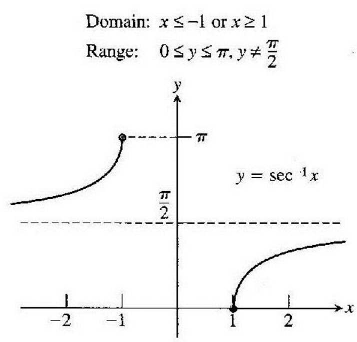

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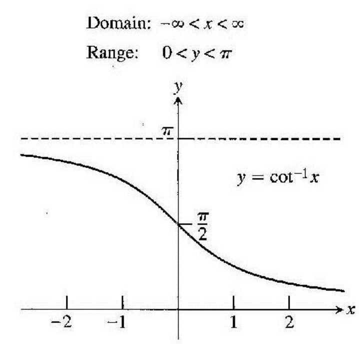

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Example 6: Tangent line to the Inverse cotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1}x$ at $x = -1$.

Point of tangency: $(-1, f(-1)) \to (-1, \cot^{-1}(-1)) \to \dfrac{\pi}{2} - \tan^{-1}(-1) = \dfrac{\pi}{2} - \left(-\dfrac{\pi}{4}\right) = \dfrac{3\pi}{4} \Rightarrow \left(-1, \dfrac{3\pi}{4}\right)$

$y' = \dfrac{-1}{1+x^2}$

At $x = -1$

$y' = \dfrac{-1}{1+(-1)^2} = \dfrac{-1}{2}$

$y_t = \dfrac{-1}{2}x + b$

$\dfrac{3\pi}{4} = \dfrac{-1}{2}(-1) + b$

$b = \dfrac{3\pi}{4} - \dfrac{1}{2} = \dfrac{3\pi - 2}{4}$

$\therefore y_t = \dfrac{-1}{2}x + \dfrac{3\pi - 2}{4}$

OR

$\cot^{-1}(-1) = ?$

$\cot ? = -1 \quad 0 < ? < \pi$

$\dfrac{1}{\tan ?} = -1$

$-1 = \tan ?$

$? = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$ (*rejected*)

Homework
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