Derivatives of Inverse Trigonometric Functions

Date:

Derivative of the Inverse Trigonometric Functions (Arc-Trigonometric)

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

Proof: Arc-sine

$y = \sin^{-1}x$

$\Rightarrow x = \sin y; \frac{-\pi}{2} \le y \le \frac{\pi}{2}$

$\frac{d}{dx}(x) = \frac{d}{dx}\sin y$

$1 = \cos y \frac{dy}{dx}$

If $y \ne \pm\frac{\pi}{2}, \frac{dy}{dx} = \frac{1}{\cos y}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

$\because x = \sin y \quad \therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Proof: Arc-cosine

$y = \cos^{-1}x$

$\Rightarrow x = \cos y, 0 \le y \le \pi$

$\frac{d}{dx}(x) = \frac{d}{dx}\cos y$

$1 = -\sin y \frac{dy}{dx}$

If $y \ne 0$ or $\pi, \frac{dy}{dx} = \frac{-1}{\sin y}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$

$\because x = \cos y \quad \therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

Proof: Arc-tangent

$y = \tan^{-1}x$

$\Rightarrow x = \tan y; \frac{-\pi}{2} < y < \frac{\pi}{2}$

$\frac{d}{dx}(x) = \frac{d}{dx}\tan y$

$1 = \sec^2 y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

$\because x = \tan y \quad \therefore \frac{dy}{dx} = \frac{1}{1+x^2}$

Recall

$\sin^2 y + \cos^2 y = 1$

$\cos y = +\sqrt{1-\sin^2 y}$

$\frac{-\pi}{2} < y < \frac{\pi}{2}$

Recall

$\sin^2 y + \cos^2 y = 1$

$\sin y = +\sqrt{1-\cos^2 y}$

$0 < y < \pi$

Recall

$\sec^2 y = 1 + \tan^2 y$

$\tan^2 y = \sec^2 y - 1$

Recall

$\csc^2 y = 1 + \cot^2 y$

$\cot^2 y = \csc^2 y - 1$

Example 1: Derivatives of Inverse Trigonometric functions

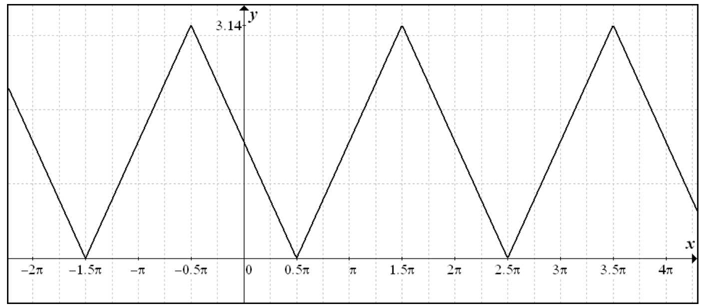
Find the derivative of the following

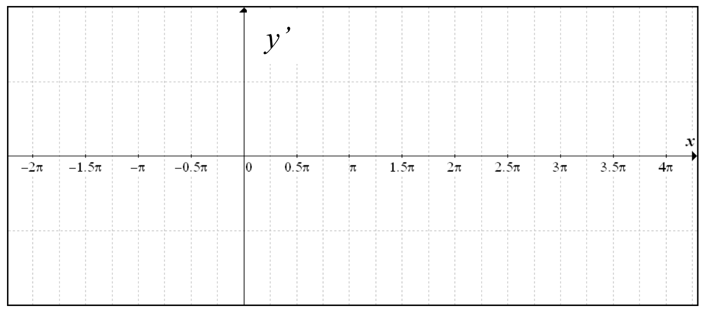
a) $y = \sin^{-1}(1-x^2)$

b) $y = x\tan^{-1}\sqrt{x}$

Example 2: Derivatives of combined Inverse-Trig Functions

Differentiate $y=\cos^{-1}(\sin x)$ and use the result to sketch the graph of the derivatives.





Example 3: Derivatives of Inverse Trigonometric functions

Find the derivative of $y=\tan^{-1}\left(\dfrac{x}{y}\right)$

Example 4: Applications of Inverse Trigonometric Functions

A particle moves along the *x*-axis so that its position at any time $t\geq 0$ is $x(t)=\tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t=16$? Distance is in *m*, time in *sec.*

Derivatives of the Other Three Inverse Trig Functions

$$\frac{d}{dx}\sec^{-1}x=\begin{cases}\frac{1}{x\sqrt{x^2-1}}, x>1\\ \frac{-1}{x\sqrt{x^2-1}}, x<-1\end{cases}$$

$$\frac{d}{dx}\csc^{-1}x=\begin{cases}\frac{-1}{x\sqrt{x^2-1}}, x>1\\ \frac{1}{x\sqrt{x^2-1}}, x<-1\end{cases}$$

$$\frac{d}{dx}\cot^{-1}x=\frac{-1}{1+x^2}$$

Proof: Arc-secant

$y=\sec^{-1}x \Rightarrow x=\sec y;$

$0\le y\le\frac{\pi}{2}$ & $\frac{\pi}{2}\le y\le\pi$

$\frac{d}{dx}(x)=\frac{d}{dx}\sec y$

$1=\sec y\tan y\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{1}{\sec y\tan y}$

$\because x=\sec y$

and $\tan y=\pm\sqrt{\sec^2 y-1}$

$\tan y=\pm\sqrt{x^2-1}$

$\therefore \frac{dy}{dx}=\frac{1}{\pm x\sqrt{x^2-1}}$

$\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$ if $x>1$

$\frac{d}{dx}\sec^{-1}x=\frac{-1}{x\sqrt{x^2-1}}$ if $x<-1$

Proof: Arc-cosecant

$y=\csc^{-1}x \Rightarrow x=\csc y;$

$0\le y\le\frac{\pi}{2}$ & $\frac{-\pi}{2}\le y\le 0$

$\frac{d}{dx}(x)=\frac{d}{dx}\csc y$

$1=-\csc y\cot y\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{-1}{\csc y\cot y}$

$\because x=\csc y$

and $\cot y=\pm\sqrt{\csc^2 y-1}$

$\cot y=\pm\sqrt{x^2-1}$

$\therefore \frac{dy}{dx}=\frac{-1}{\pm x\sqrt{x^2-1}}$

$\frac{d}{dx}\csc^{-1}x=\frac{-1}{x\sqrt{x^2-1}}$ if $x>1$

$\frac{d}{dx}\csc^{-1}x=\frac{1}{x\sqrt{x^2-1}}$ if $x<-1$

Proof: Arc-cotangent

$y=\cot^{-1}x \Rightarrow x=\cot y;$

$0\le y\le\pi$

$\frac{d}{dx}(x)=\frac{d}{dx}\cot y$

$1=-\csc^2 y\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{-1}{\csc^2 y}$

$\because \csc^2 y=1+\cot^2 y$

$\therefore \frac{dy}{dx}=\frac{-1}{1+\cot^2 y}$

$\because x=\cot y$

$\therefore \frac{dy}{dx}=\frac{-1}{1+x^2}$

Example 5: Derivative of the Co-Inverse Trig Functions

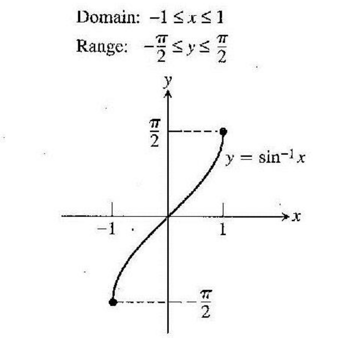
Find the derivative of $\sec^{-1}(5x^4)$

Inverse Function-Inverse Cofunction Identites (Proofs of the identities can be determined graphically)

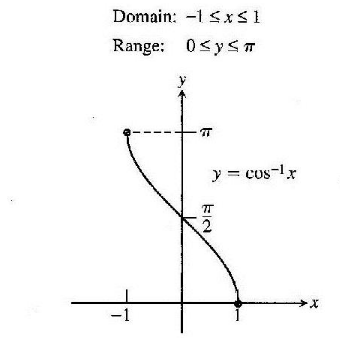
$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x \qquad \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x \qquad \csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x$$

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \qquad \csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

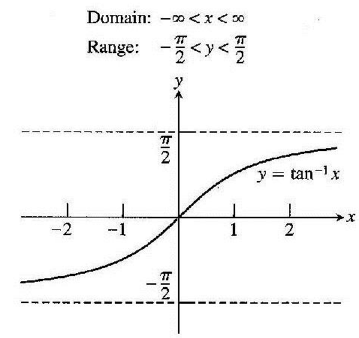
Recall:



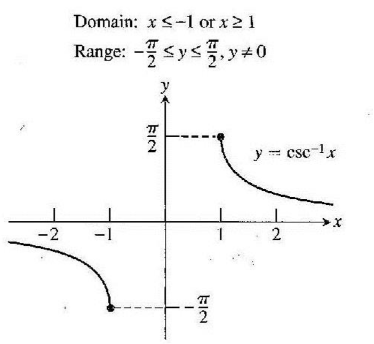
(a)



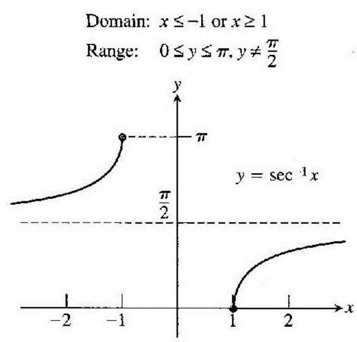
(b)



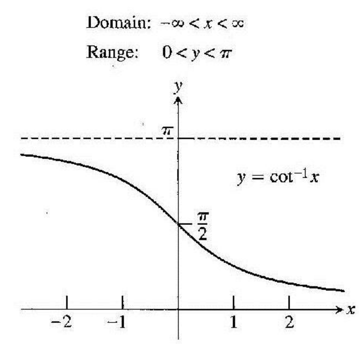
(c)



(d)



(e)



(f)

Example 6: Tangent line to the Inverse cotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1}x$ at $x = -1$.

Homework
P. 170 #1-29