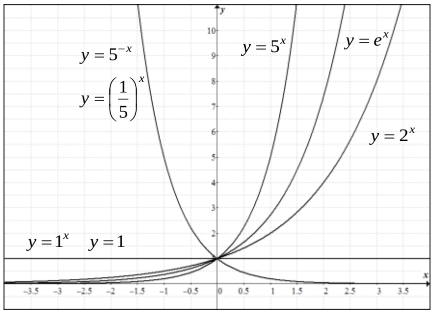
Derivatives of Exponential & Logarithmic Functions

Date:

Definition of *e* (Natural Exponential Number)

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} \approx 2.718281828459......$$



Properties of $y = e^x$ and $y = \ln x$

Recall the logarithmic function is the inverse of the exponential function.

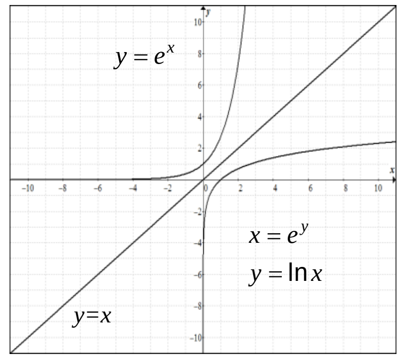
$\because y = \log_b x \Leftrightarrow b^y = x$ which is the inverse of $b^x = y$,

$\therefore y = \log_e x \Leftrightarrow e^y = x$ which is the inverse of $y = e^x$.

The function $y = \log_e x$ can be written as $y = \ln x$ and is called the **natural logarithm function**.

$y = \log_b x \Leftrightarrow b^y = x$

$y = \ln x \Leftrightarrow e^y = x$



$y = e^x$	$y = \ln x$
• Domain is $x \in R$ • Range is $y \in R \mid y > 0$ • y-intercept at 1 • $e^{\ln x} = x, x > 0$ • Horizontal Asymptote $y = 0$.	• Domain is $x \in R \mid x > 0$ • Range is $y \in R$ • x-intercept at 1 • $\ln e^x = x, x \in R$ • Vertical Asymptote $x = 0$.

Derivative of $f(x) = e^x$

Rule (1): If $y = e^x$, then $\frac{dy}{dx} = e^x$. **Rule (2): If** $f(x) = e^{g(x)}$**, then** $f'(x) = e^{g(x)}g'(x)$

Example 1: Derivative of Exponential functions

Differentiate

a) $y = e^{4x^2-5x+3}$

$y' = e^{4x^2-5x+3}\cdot(8x-5)$

$= (8x-5)e^{4x^2-5x+3}$

b) $y = 5e^{3x^2-2x+7}$

$y' = 5e^{3x^2-2x+7}\cdot(6x-2)$

$= 10(3x-1)e^{3x^2-2x+7}$

c) $h(t) = \dfrac{e^{3t}}{1-e^{3t}}$

$h'(t) = \dfrac{3e^{3t}(1-e^{3t}) - e^{3t}(-3e^{3t})}{(1-e^{3t})^2}$

$= \dfrac{3e^{3t}(1-e^{3t}) + 3(e^{3t})^2}{(1-e^{3t})^2} = \dfrac{3e^{3t}\lfloor(1-e^{3t}) + e^{3t}\rfloor}{(1-e^{3t})^2}$

$= \dfrac{3e^{3t}}{(1-e^{3t})^2}$

Example 2: Connecting the derivative of an exponential function to slope of tangent.

Find all points at which the tangent to the curve defined by $y = 3x^2e^{4x}$ is horizontal.

$y' = (6x)(e^{4x}) + (3x^2)(e^{4x})(4)$

$= (6x)(e^{4x}) + (12x^2)(e^{4x})$

$= 6xe^{4x}(1+2x)$

Slope = y' (Horizontal tangent when m = 0)

$y' = 0$

$0 = 6xe^{4x}(1+2x)$

$x = 0 \quad x = -0.5$

$\therefore$ Points (0, f(0)) & (-0.5, f(-0.5))

➔ (0, 0) & (-0.5, $3/4e^2$)

Definition of Natural Logarithm

Natural logarithm is the logarithm to the base *e*. $\log_e x \Leftrightarrow \ln x$

Basic Properties

1) $\log_e 1 = 0 \quad \Leftrightarrow \ln 1 = 0$

2) $\log_e e = 1 \quad \Leftrightarrow \ln e = 1$

3) $\log_e e^x = x \quad \Leftrightarrow \ln e^x = x$

4) $e^{\log_e x} = x \quad \Leftrightarrow e^{\ln x} = x$

Laws of Logarithm

1) $\log_e xy = \log_e x + \log_e y \quad \Leftrightarrow \ln xy = \ln x + \ln y$ **(Product Law)**

2) $\log_e \dfrac{x}{y} = \log_e x - \log_e y \quad \Leftrightarrow \ln \dfrac{x}{y} = \ln x - \ln y$ **(Quotient Law)**

3) $\log_e x^p = p \log_e x \quad \Leftrightarrow \ln x^p = p \ln x$ **(Power Law)**

Derivative of *y = lnx*

If *y* = ln *x*, then $\dfrac{dy}{dx} = \dfrac{1}{x}$**.**

If *y* = ln *u*, then $\dfrac{dy}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$.

Proof :

$y = \ln x \Leftrightarrow e^y = x$

$\dfrac{d}{dx}(\ln x) \Rightarrow \dfrac{d}{dx}e^y = \dfrac{d}{dx}x$

$e^y \dfrac{dy}{dx} = 1 \qquad \dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$

Example 3: Derivatives of Natural Logarithmic functions

Differentiate

a) $y = \ln(5x^2 + 3)$

$y' = \dfrac{1}{5x^2+3}(10x)$

$= \dfrac{10x}{5x^2+3}$

b) $y = \ln(5x^2+3)^4$

$y = 4\ln(5x^2+3)$

$y' = 4\dfrac{(10x)}{5x^2+3}$

$= \dfrac{40x}{5x^2+3}$

c) $y = \left[\ln(5x^2+3)\right]^4$

$y' = 4\left[\ln(5x^2+3)\right]^3 \dfrac{10x}{5x^2+3}$

$= \dfrac{40x\left[\ln(5x^2+3)\right]^3}{5x^2+3}$

Example 4: Derivatives of Natural Logarithmic functions

Differentiate

a) $h(u) = e^{\sqrt{u}} \ln \sqrt{u}$

$h(u) = e^{u^{\frac{1}{2}}} \cdot \dfrac{1}{2}\ln u$

$h'(u) = e^{\sqrt{u}} \cdot \dfrac{1}{2}u^{\frac{-1}{2}} \cdot \dfrac{1}{2}\ln u + e^{\sqrt{u}} \cdot \dfrac{1}{2} \cdot \dfrac{1}{u}$

$= \dfrac{1}{4\sqrt{u}} \cdot e^{\sqrt{u}} \cdot \ln u + \dfrac{1}{2u} \cdot e^{\sqrt{u}}$

$= \dfrac{e^{\sqrt{u}}}{4u}\left(\sqrt{u}\ln u + 2\right)$

b) $y = \dfrac{\ln x}{x^2}$

$y' = \dfrac{\left(\frac{1}{x}\right)(x^2) - (\ln x)(2x)}{x^4}$

$= \dfrac{x - 2x\ln x}{x^4}$

$= \dfrac{1 - 2\ln x}{x^3}$

Example 5: Derivatives of Natural Logarithmic functions with Laws

Differentiate a) $f(x)=\ln\sqrt{\dfrac{x^2+1}{x-1}}$

$f(x)=\ln\left(\dfrac{x^2+1}{x-1}\right)^{\frac{1}{2}}=\dfrac{1}{2}\ln\left(\dfrac{x^2+1}{x-1}\right)$

$=\dfrac{1}{2}\left[\ln\left(x^2+1\right)-\ln(x-1)\right]$

$f'(x)=\dfrac{1}{2}\left(\dfrac{2x}{x^2+1}-\dfrac{1}{x-1}\right)$

$=\dfrac{1}{2}\left[\dfrac{2x(x-1)-(x^2+1)}{\left(x^2+1\right)(x-1)}\right]$

$=\dfrac{1}{2}\left[\dfrac{2x^2-2x-x^2-1}{\left(x^2+1\right)(x-1)}\right]$

$=\dfrac{x^2-2x-1}{2\left(x^2+1\right)(x-1)}$

b) $f(x)=\ln\left[(3x+1)(2x-5)\right]$

$f(x)=\ln(3x+1)+\ln(2x-5)$

$f'(x)=\dfrac{3}{3x+1}+\dfrac{2}{2x-5}$

$=\dfrac{3(2x-5)+2(3x+1)}{(3x+1)(2x-5)}$

$=\dfrac{6x-15+6x+2}{(3x+1)(2x-5)}$

$=\dfrac{12x-13}{(3x+1)(2x-5)}$

Example 6: Equation of tangent to the Natural logarithmic function

Find the equation of the tangent to the curve defined by $y=\ln\left(1+2e^{-x}\right)$ at the point where $x=0$.

$m=y'=\dfrac{-2e^{-x}}{1+2e^{-x}}$

At $x=0$

$m=\dfrac{-2}{1+2}=\dfrac{-2}{3}$

Point of tangency $(0,\ln 3)$

$y_t=\dfrac{-2}{3}x+b$

$\ln 3=\dfrac{-2}{3}(0)+b$

$b=\ln 3$

$\therefore y_t=\dfrac{-2}{3}x+\ln 3$

Example 7: Equation of tangent to the Natural logarithmic function

Find the equation of the tangent to the curve defined by $y=e^x$ that is perpendicular to the line defined by $4x+y=1$.

$y=-4x+1$

$m=-4 \quad m_{\perp}=\dfrac{1}{4}$

For $y=e^x$

$m_t=y'=e^x$

$\dfrac{1}{4}=e^x$

$\ln\left(\dfrac{1}{4}\right)=\ln e^x$

$\ln 1-\ln 4=x$

$x=-\ln 4$

Point of tangency $(-\ln 4, e^{-\ln 4})\rightarrow(-\ln 4,\dfrac{1}{4})$

$y_t=\dfrac{1}{4}x+b$

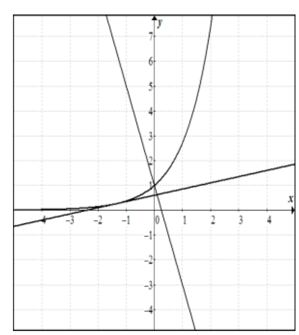
$\dfrac{1}{4}=\dfrac{1}{4}(-\ln 4)+b$

$\dfrac{1}{4}=-\dfrac{1}{4}\ln 4+b$

$b=\dfrac{1}{4}+\dfrac{1}{4}\ln 4$

$y_t=\dfrac{1}{4}x+\dfrac{1}{4}+\dfrac{1}{4}\ln 4$

$x-4y+1+\ln 4=0$



Derivative of General Exponential functions

$$y = b^{g(x)} \qquad \frac{1}{y}\frac{dy}{dx} = g'(x)\ln b + g(x)(0)$$
$$\ln y = \ln b^{g(x)}$$
$$\ln y = g(x)\ln b \qquad \frac{dy}{dx} = y \cdot \ln b \cdot g'(x)$$
$$\frac{dy}{dx} = b^{g(x)} \cdot \ln b \cdot g'(x)$$

Derivative of General Exponential functions

For $y = b^{g(x)}$ $\qquad y' = b^{g(x)} \ln b \bullet g'(x)$

Example 8: Derivative of general exponential functions

Differentiate

a) $y = 5^{3x}$
$$y' = 5^{3x} \cdot \ln 5 \cdot 3$$
$$= 5^{3x} \cdot 3\ln 5$$

b) $y = 4^{3x^2-5x+1}$
$$y' = 4^{3x^2-5x+1} \cdot \ln 4 \cdot (6x-5)$$
$$= 4^{3x^2-5x+1} \cdot (6x-5) \cdot \ln 4$$

c) $y = 2x^4 \cdot 4^{2x} + e^{\ln 5}$
$$y = 2x^4 \cdot 4^{2x} + 5$$
$$y' = 8x^3 \cdot 4^{2x} + 2x^4 \cdot 4^{2x} \cdot \ln 4 \cdot 2$$
$$= 8x^3 \cdot 4^{2x} + 4x^4 \cdot 4^{2x} \cdot \ln 4$$
$$= 4x^3 \cdot 4^{2x}(2 + x\ln 4)$$
or
$$= 4x^3 \cdot 4^{2x}(2 + x\ln 2^2)$$
$$= 4x^3 \cdot 4^{2x}(2 + 2x\ln 2)$$
$$= 8x^3 \cdot 4^{2x}(1 + x\ln 2)$$

d) $f(x) = \dfrac{4\ln\sqrt{3x}}{4^{\ln 8x}}$
$$f(x) = \frac{4\ln(3x)^{\frac{1}{2}}}{4^{\ln 8x}} = \frac{2\ln(3x)}{4^{\ln 8x}}$$
$$f'(x) = \frac{2 \cdot \frac{3}{3x} \cdot 4^{\ln 8x} - 2\ln(3x) \cdot 4^{\ln 8x} \cdot \ln 4 \cdot \frac{8}{8x}}{\left(4^{\ln 8x}\right)^2}$$
$$= \frac{2 \cdot \frac{1}{x} \cdot 4^{\ln 8x} - 2\ln(3x) \cdot 4^{\ln 8x} \cdot \ln 4 \cdot \frac{1}{x}}{\left(4^{\ln 8x}\right)^2}$$
$$= \frac{\frac{2}{x} \cdot 4^{\ln 8x}(1 - \ln 3x \cdot \ln 4)}{\left(4^{\ln 8x}\right)^2}$$
$$= \frac{2 \cdot (1 - \ln 3x \cdot \ln 4)}{x\left(4^{\ln 8x}\right)} \text{ or } \frac{2 \cdot (1 - 2 \cdot \ln 3x \cdot \ln 2)}{x\left(4^{\ln 8x}\right)}$$

Example 9: Solving a problem involving an exponential model

A biologist is studying the increase in the population of a particular insect in a provincial park. The population triples every week. Assume the population continues to increase at this rate. Initially there are 100 insects.

a) Write an equation to represent the number of insects in *t* weeks.
b) Determine the number of insects present after 4 weeks.
c) How fast is the number of insects increasing
   i) when they are initially discovered?
   ii) at the end of 4 weeks?

Let N be number of insects
t be number of weeks

a) $N(t) = (100)3^t$

b) $$N(4) = (100)3^4$$
$$= (100)(81)$$
$$= 8100$$

There are 8100 insects after 4 weeks

c) $$N(t) = (100)3^t$$
$$N'(t) = (100)(3^t)\ln 3$$

i) Initially discovered ($t = 0$)

$$N'(0) = (100)(3^0)\ln 3$$
$$= 100\ln 3$$
$$\approx 110$$

Therefore the insects are increasing at a rate of approx. 110 per week at the beginning of the week.

ii) At the end of 4 weeks

$$N'(4) = (100)(3^4)\ln 3$$
$$= 8100\ln 3$$
$$\approx 8899$$

Therefore the insects are increasing at a rate of approx. 8899 per week at the end of the 4 weeks.

Recall: Change of base

$$\log_a b \Leftrightarrow \frac{\log_e b}{\log_e a} \Leftrightarrow \frac{\ln b}{\ln a}$$

Derivative of general logarithmic function

For $y = \log_a g(x)$ $\qquad y' = \dfrac{g'(x)}{g(x)\ln a}$

Basic Properties

1) $\log_b 1 = 0$
2) $\log_b b = 1$
3) $\log_b b^x = x$
4) $b^{\log_b x} = x$

Laws of Logarithm

1) $\log_b xy = \log_b x + \log_b y$
2) $\log_b \dfrac{x}{y} = \log_b x - \log_b y$
3) $\log_b x^p = p\log_b x$

Proof

$$y = \log_a g(x) \Rightarrow \frac{\log_e g(x)}{\log_e a} \Rightarrow \frac{\ln g(x)}{\ln a}$$

$$y' = \frac{\dfrac{g'(x)}{g(x)} \cdot \ln a - \ln g(x) \cdot (0)}{(\ln a)^2}$$

$$= \frac{\dfrac{g'(x)}{g(x)}}{(\ln a)} = \frac{g'(x)}{g(x)\ln a}$$

Example 10: Derivative of general logarithmic functions

Differentiate

a) $y=\log_2\left(6x^2+4x-7\right)$

$$y'=\frac{12x+4}{\left(6x^2+4x-7\right)\ln 2}$$
$$=\frac{4(3x+1)}{\left(6x^2+4x-7\right)\ln 2}$$

b) $y=\log_2(4x^2+3x-1)^5$

$$y=5\log_2\left(4x^2+3x-1\right)$$
$$y'=\frac{5(8x+3)}{\left(4x^2+3x-1\right)\ln 2}$$
$$=\frac{5(8x+3)}{(4x-1)(x+1)\ln 2}$$

c) $y=\dfrac{-2\log_5\left(25^{3x}\right)}{5^{\log_5(\ln x)}}$

$$y=\frac{-2\log_5\left(5^2\right)^{3x}}{5^{\log_5(\log_e x)}}=\frac{-2\log_5\left(5^{6x}\right)}{\log_e x}=\frac{-12x}{\ln x}$$
$$y'=\frac{(-12)(\ln x)-(-12x)\left(\frac{1}{x}\right)}{(\ln x)^2}$$
$$=\frac{-12\ln x+12}{(\ln x)^2}$$
$$=\frac{-12(\ln x-1)}{(\ln x)^2}$$

d) $y=\log_3\sqrt{\left(\dfrac{2+x}{x-2}\right)}$

$$y=\log_3\left(\frac{2+x}{x-2}\right)^{\frac{1}{2}}$$
$$=\frac{1}{2}\left[\log_3(2+x)-\log_3(x-2)\right]$$
$$y'=\frac{1}{2}\left[\frac{1}{(2+x)\ln 3}-\frac{1}{(x-2)\ln 3}\right]$$
$$=\frac{1}{2}\left[\frac{(x-2)-(x+2)}{(x+2)(x-2)\ln 3}\right]$$
$$=\frac{-4}{2(x+2)(x-2)\ln 3}$$
$$=\frac{-2}{(x+2)(x-2)\ln 3}$$

Homework:
P. 178 #1-42, 51, 53

Cal & Vec (Optional)
P. 232 #2 – 13, 15
P. 575 #3,4,5ab,6,7a,8, 9a,10-13
P. 240 # 1 – 6, 7ab, 8
P. 578 # 1, 2, 3a, 4acef, 5, 7, 9a