Logarithmic Differentiation Date:

Logarithmic Differentiation

The derivatives of most functions involving exponential and logarithmic expressions can be determined by using the methods that we have developed. A function $y = x^x$ poses new problems, however. The power rule cannot be used because the exponent is not a constant. The method of determining the derivative of an exponential function also cannot be used because the base is not a constant. In such cases we are using the **logarithmic differentiation**.

Steps in Logarithmic Differentiation

1) Take logarithms of both sides of an equation $y = f(x)$.
2) Differentiate implicitly with respect to *x*.
3) Solve the resulting equation for *y*'.

Example 1: Determine the derivative of a function using logarithmic differentiation

Determine $\frac{dy}{dx}$ for the function $y = x^x, x > 0$.

$$\ln(y) = \ln(x^x)$$
$$\ln y = x\ln x$$
$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln x$$
$$\frac{1}{y}\frac{dy}{dx} = 1\ln x + x\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = y(\ln x + 1)$$
$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Example 2: Determine the derivative of a function using logarithmic differentiation

For $y = (x^2+3)^x$, $\frac{dy}{dx}$.

$$\ln(y) = \ln(x^2+3)^x$$
$$\ln y = x\ln(x^2+3)$$
$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln(x^2+3)$$
$$\frac{1}{y}\frac{dy}{dx} = 1\ln(x^2+3) + x\left(\frac{2x}{x^2+3}\right)$$
$$\frac{dy}{dx} = y\left[\ln(x^2+3) + \frac{2x^2}{x^2+3}\right]$$
$$\frac{dy}{dx} = (x^2+3)^x\left[\ln(x^2+3) + \frac{2x^2}{x^2+3}\right]$$

OR

$$\frac{dy}{dx} = (x^2+3)^x\left[\frac{(x^2+3)\ln(x^2+3) + 2x^2}{x^2+3}\right]$$
$$\frac{dy}{dx} = (x^2+3)^{x-1}\left[(x^2+3)\ln(x^2+3) + 2x^2\right]$$

Example 3: Determine the derivative of a function using logarithmic differentiation

Given $y=\dfrac{\left(x^4+1\right)\sqrt{x+2}}{\left(2x^2+2x+1\right)}$, determine $\dfrac{dy}{dx}$ at x = -1.

$$\ln(y)=\ln\left[\frac{\left(x^4+1\right)\sqrt{x+2}}{\left(2x^2+2x+1\right)}\right]$$

$$\ln y=\ln\left(x^4+1\right)+\ln(x+2)^{\frac{1}{2}}-\ln\left(2x^2+2x+1\right)$$

$$\ln y=\ln\left(x^4+1\right)+\frac{1}{2}\ln(x+2)-\ln\left(2x^2+2x+1\right)$$

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left[\ln\left(x^4+1\right)+\frac{1}{2}\ln(x+2)-\ln\left(2x^2+2x+1\right)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{4x^3}{x^4+1}+\frac{1}{2(x+2)}-\frac{4x+2}{2x^2+2x+1}$$

$$\frac{dy}{dx}=y\left[\frac{4x^3}{x^4+1}+\frac{1}{2(x+2)}-\frac{4x+2}{2x^2+2x+1}\right]$$

$$\frac{dy}{dx}=\frac{\left(x^4+1\right)\sqrt{x+2}}{\left(2x^2+2x+1\right)}\left[\frac{4x^3}{x^4+1}+\frac{1}{2(x+2)}-\frac{4x+2}{2x^2+2x+1}\right]$$

At x = -1

$$\frac{dy}{dx}=\frac{(1+1)\sqrt{1}}{(2-2+1)}\left[\frac{-4}{1+1}+\frac{1}{2(-1+2)}-\frac{-4+2}{2-2+1}\right]$$

$$\frac{dy}{dx}=2\left(-2+\frac{1}{2}+2\right)=1$$

Optional
Not necessary in this question

$$\frac{dy}{dx}=\frac{\left(x^4+1\right)\sqrt{x+2}}{\left(2x^2+2x+1\right)}\left[\frac{4x^3}{x^4+1}+\frac{1}{2(x+2)}-\frac{4x+2}{2x^2+2x+1}\right]$$

$$=\frac{\left(x^4+1\right)\sqrt{x+2}}{\left(2x^2+2x+1\right)}\left[\frac{4x^3(2)(x+2)\left(2x^2+2x+1\right)+\left(x^4+1\right)\left(2x^2+2x+1\right)-(4x+2)(2)(x+2)\left(x^4+1\right)}{2\left(x^4+1\right)(x+2)\left(2x^2+2x+1\right)}\right]$$

$$=\frac{4x^3(2)(x+2)\left(2x^2+2x+1\right)+\left(x^4+1\right)\left(2x^2+2x+1\right)-(4x+2)(2)(x+2)\left(x^4+1\right)}{2\sqrt{(x+2)}\left(2x^2+2x+1\right)^2}$$

$$=\frac{\left(8x^4+16x^3\right)\left(2x^2+2x+1\right)+\left(x^4+1\right)\left(2x^2+2x+1\right)-(4x+2)(2x+4)\left(x^4+1\right)}{2\sqrt{(x+2)}\left(2x^2+2x+1\right)^2}$$

$$=\frac{10x^6+30x^5+33x^4+16x^3-6x^2-18x-7}{2\sqrt{(x+2)}\left(2x^2+2x+1\right)^2}$$

Homework: P. 582
1-9, 12, 13