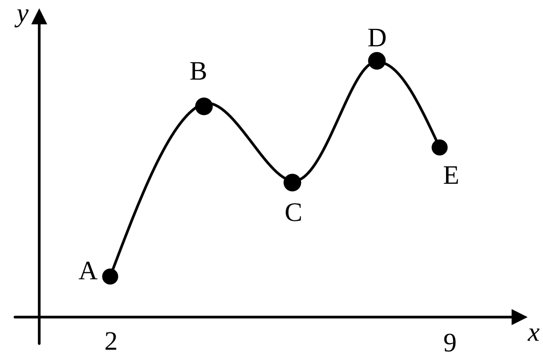
Extreme Values of Functions (First Derivative Test)

Date:



Absolute maximum and minimum points

- The highest and the lowest points in the figure.
- The highest and the lowest points in the domain.
- In the figure, D = absolute max. point; A = absolute min. point
- Also known as Global extreme values.

Local maximum and minimum points

- The turning points.
- The highest and the lowest points in a particular interval.
- In the figure, B & D = local max. points; C = local min. point
- Also known as Relative extreme values.

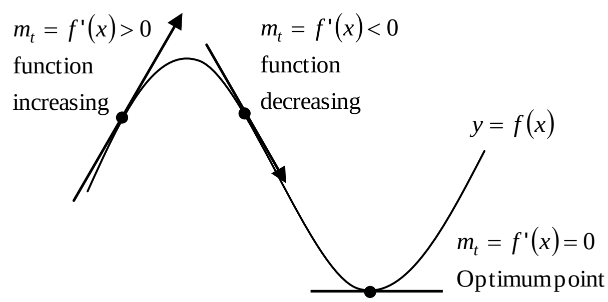
End points

- The extreme points in the figure, usually given in the form $x_1 \le x \le x_2$.
- In the figure, A & E = extreme points ($2 \le x \le 9$).

Critical numbers of *f '*(*x*)

- They are the values of *x* when
  i) $f'(x) = 0$, or ii) $f'(x)$ does not exist (usually we set the denominator of *f'*(*x*)=0 and solve for *x*)
- Critical numbers generate the *x*-coordinates of the turning points.

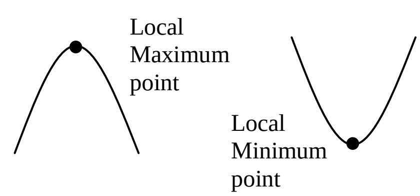
General rules (First Derivative Test)



1) The function is increasing if $f'(x) > 0$.
   The function is decreasing if $f'(x) < 0$.

2) The critical numbers are values of *x* when
   i) $f'(x) = 0$, or
   ii) $f'(x)$ DNE (usually we set the denominator of $f'(x) = 0$.

3) The Local maximum/minimum points are turning points occurring between increasing and decreasing of function or vice versa.



Example 1: Extreme values of a function

Find the extreme values of the function $f(x) = x^3 - 12x - 3$ on the interval $-3 \le x \le 5$.

$$f'(x) = 3x^2 - 12$$
$$= 3(x^2 - 4)$$
$$= 3(x-2)(x+2)$$

Critical numbers for $f'(x)$

i) $f'(x) = 0$ when $x = 2$ and $x = -2$

ii) $f'(x)$ undefined: None

Test the Critical numbers and the Endpoints

$f(2) = (2)^3 - 12(2) - 3 = 8 - 24 - 3 = -19$ (Abs.Min)

$f(-2) = (-2)^3 - 12(-2) - 3 = -8 + 24 - 3 = 13$

$f(-3) = (-3)^3 - 12(-3) - 3 = -27 + 36 - 3 = 6$

$f(5) = (5)^3 - 12(5) - 3 = 125 - 60 - 3 = 62$ (Abs.Max)

Absolute maximum value is 62 occurs when *x* = 5

Absolute minimum value is -19 occurs when *x* = 2

Example 2: Increasing and decreasing function & its local extreme values

Given $f(x) = x^3 - 6x^2$.

a) Find the critical numbers.
b) Find the intervals of increase and decrease.
c) Find the local maximum and local minimum points.
d) Sketch the graph

$f'(x) = 3x^2 - 12x$

$= 3x(x-4)$

Critical Numbers of $f'(x)$

1) $f'(x) = 0$ when $x = 0, x = 4$

2) $f'(x)$ und : None

$f(x) = x^2(x-6)$

Domain of $x, x \in R$

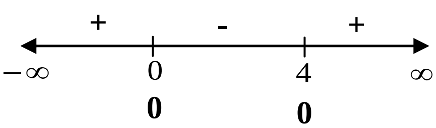
Critical Numbers of $f(x)$

1) $f(x) = 0$ when $x^2(x-6) = 0$

$x = 0$ and $x = 6$ (x - intercepts)

2) $f(x)$ und : None

y - intercept $f(0) = 0$

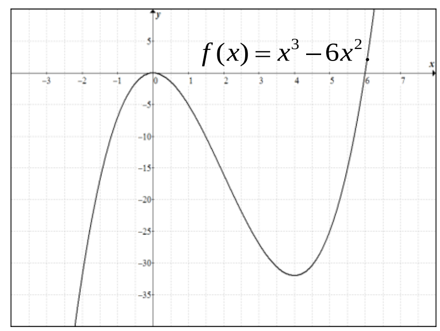


Function increasing in intervals $(-\infty, 0)$ & $(4, \infty)$

Function decreasing in interval $(0, 4)$

Local Max at $x = 0$ $(0, f(0)) \to (0,0)$

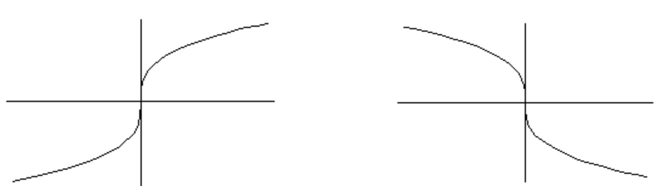
Local Min at $x = 4$ $(4, f(4)) \to (4,-32)$



Vertical Tangents and Cusps

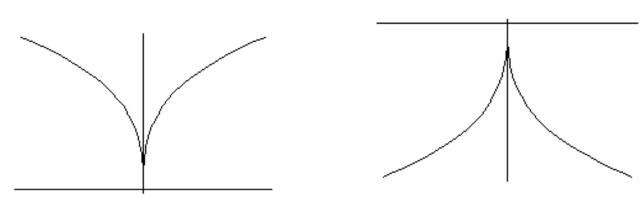
There are four possibilities for unbounded behavior of a derivative $f'(x)$ around a given real number ***C***, all of them occur when the critical numbers x are obtained from $f'(x)$ **DNE.** The four possible cases are:

Cases 1 & 2: Vertical Tangents (limits same in sign)



$\lim_{x \to C^-} f'(x) = \lim_{x \to C^+} f'(x) = +\infty$ $\lim_{x \to C^-} f'(x) = \lim_{x \to C^+} f'(x) = -\infty$

Cases 3 & 4: Cusps (limits differ in sign)



$\lim_{x \to C^-} f'(x) = -\infty$ $\lim_{x \to C^+} f'(x) = +\infty$ $\lim_{x \to C^-} f'(x) = +\infty$ $\lim_{x \to C^+} f'(x) = -\infty$

Example 3: Vertical Tangent or Cusps

Repeat Example 1 for $f(x) = (x+2)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(x+2)^{\frac{-1}{3}}$

$= \frac{2}{3(x+2)^{\frac{1}{3}}}$

Critical Numbers of $f'(x)$

1) $f'(x) = 0$: *None*

2) $f'(x)$ und : $x = -2$

Domain of $x, x \in R$

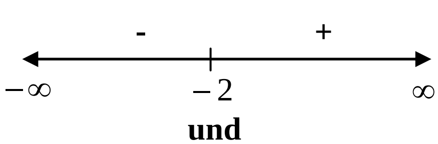
Critical Numbers of $f(x)$

1) $f(x) = 0$ when $(x+2)^{\frac{2}{3}} = 0$

$x = -2$ (x-intercepts)

2) $f(x)$ und : None

y-intercept $f(0) = 2^{\frac{2}{3}} \approx 1.6$

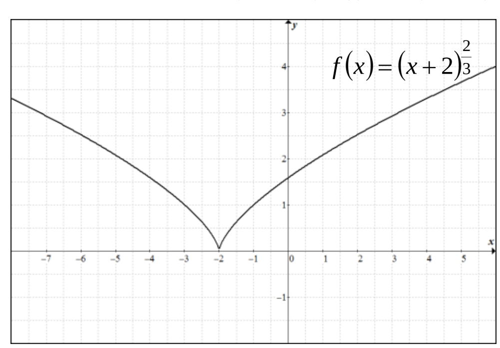


Function increasing in interval $(-2, \infty)$

Function decreasing in interval $(-\infty, -2)$

Local Max : None

Local Min at $x = -2$ $(-2, f(-2)) \to (-2, 0)$



Example 4: Determine the function

Find values for *a*, *b*, and *c* such that the graph of $f(x) = ax^2 + bx + c$ has a relative maximum at (3, 12) and crosses the *y*-axis at (0, 1).

$f'(x) = 2ax + b$
$f'(3) = 0$
$0 = 2a(3) + b$
$0 = 6a + b$

$f(0) = 1$
$1 = a(0^2) + b(0) + c$
$c = 1$

$f(3) = 12$
$12 = a(3^2) + b(3) + c$
$12 = 9a + 3b + c$
$12 = 9a + 3b + 1$
$9a + 3b = 11$

$6a + b = 0$
$9a + 3b = 11$

$18a + 3b = 0$
$9a + 3b = 11$
$-------$
$9a = -11$
$a = \frac{-11}{9}$

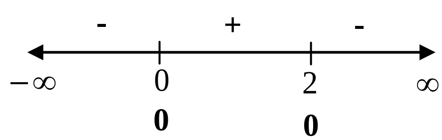
$0 = 6\left(\frac{-11}{9}\right) + b$
$b = \frac{22}{3}$

$f(x) = \frac{-11}{9}x^2 + \frac{22}{3}x + 1$

Example 5: Increasing and decreasing function & its local extreme values

Given: $f(x) = x^2 e^{-x}$

a) Determine the domain
b) Find the intervals of increase and decrease.
c) Find the local maximum and local minimum points.
d) Determine the intercepts
e) Sketch the graph



$f'(x) = (2x)(e^{-x}) + (x^2)(-e^{-x})$
$= xe^{-x}(2 - x)$
$= \frac{x(2-x)}{e^x}$

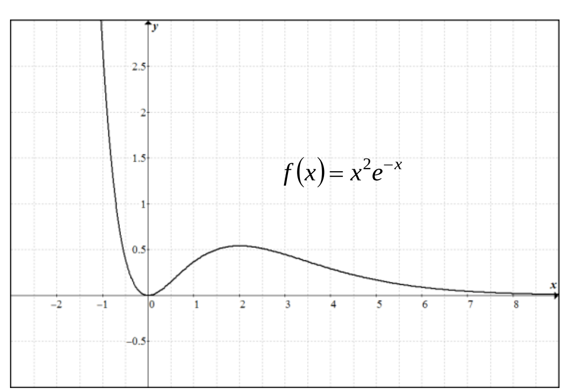
Critical Numbers of $f'(x)$
1) $f'(x) = 0 : x = 0, x = 2$
2) $f'(x)$ und : *None*

Function increasing in interval (0, 2)
Function decreasing in intervals $(-\infty, 0)$ and $(2, \infty)$
Local Max at $x = 2$ $(2, f(2)) \rightarrow \left(2, \frac{4}{e^2}\right) \rightarrow (2, 0.54)$
Local Min at $x = 0$ $(0, f(0)) \rightarrow (0, 0)$

Domain of $x, x \in R$
Range : (Optional), $y \geq 0$
Critical Numbers of $f(x)$
1) $f(x) = 0$ when $x = 0$
2) $f(x)$ und : None
y-intercept $f(0) = 0$

HA : $y = 0$
Cross over at $x = 0$

Recall : $y = \frac{ax^m}{bx^n}$
HA : $y = 0$ if $m < n$
HA : none if $m > n$
HA : $y = \frac{a}{b}$ if $m = n$
OA : if $m - n = 1$



<u>Example 6: Determine the maximum and minimum values of the trig. function</u>

Determine the maximum and minimum values of the function $y=\cos^2 x$ on the interval $x\in[0,2\pi]$.

$f'(x)=2(\cos x)(-\sin x)$

$=-2\sin x\cos x$

$=-\sin 2x$

Critical numbers of $f'(x)$

i) $f'(x)=0$ when

$-\sin 2x=0$ $\quad\sin r=0$

$\sin 2x=0$ $\quad 2x=r=0,\pi,2\pi,\ldots$

$r=2x$ $\quad x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi$

ii) $f'(x)$ undefined: Never

Testing the endpoints and critical numbers

$f(0)=\cos^2(0)=1$

$f(2\pi)=\cos^2(2\pi)=1$

$f\left(\frac{\pi}{2}\right)=\cos^2\left(\frac{\pi}{2}\right)=0$

$f(\pi)=\cos^2(\pi)=1$

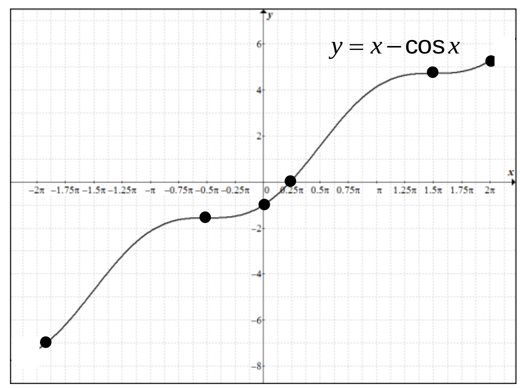
$f\left(\frac{3\pi}{2}\right)=\cos^2\left(\frac{3\pi}{2}\right)=0$

$\therefore$ Maximum value is 1 when $x=0,\pi,2\pi$ or $n\pi, n\in I$.
Minimum value is 0 when $x=\frac{\pi}{2},\frac{3\pi}{2}$ or $\frac{\pi}{2}+n\pi, n\in I$.

<u>Example 7: Increasing and decreasing function & its local extreme values</u>

Given: $f(x)=x-\cos x$, $-2\pi\le x\le 2\pi$

a) Find the intervals of increase and decrease.
b) Find the local maximum and local minimum points.
c) Sketch the graph



$f'(x)=1-(-\sin x)$

$=1+\sin x$

Critical numbers of $f'(x)$

1) $f'(x)=0$

$\because -1\le\sin x\le 1$

$\therefore f'(x)\ge 0$

2) $f'(x)$ und : None

Function always increasing.

No Local Max/Min

Absolute values :

Abs. Min: $f(-2\pi)=-2\pi-\cos(-2\pi)=-2\pi-1$

Abs. Max: $f(2\pi)=2\pi-\cos(2\pi)=2\pi-1$

Domain of $x, x\in R$

Range : (Optional), $y\in R$

Critical Numbers of $f(x)$

1) $f(x)=0$ when

$x-\cos x=0$

by trial, $x\approx 0.739$

(not required)

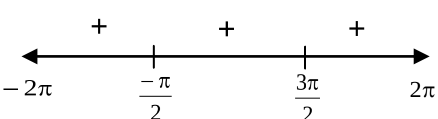
2) $f(x)$ und : None

y-intercept

$f(0)=0-\cos 0=-1$

x	cosx
0.5	0.877583
0.4	0.921061
0.6	0.825336
0.7	0.764842
0.8	0.696707
0.75	0.731689
0.74	0.738469
0.73	0.745174
0.739	0.739142

Method 2 for critical numbers of $f'(x)$



1) $f'(x)=0$

$\sin x=-1$

$x=\frac{-\pi}{2}\ \&\ \frac{3\pi}{2}$

2) $f'(x)$ und : None

Horizontal tangent **0** **0**

$x=\frac{-\pi}{2}$ $\quad\left(\frac{-\pi}{2},\frac{-\pi}{2}\right)\to\left(\frac{-\pi}{2},-1.6\right)$

$x=\frac{3\pi}{2}$ $\quad\left(\frac{3\pi}{2},\frac{3\pi}{2}\right)\to\left(\frac{3\pi}{2},4.7\right)$

Homework:
P. 193 #2-30, 35-41
<u>Cal & Vectors (Optional)</u>
P. 169 #1,3,5,7,8
P. 178 #5,7,8,9,11,15
P. 233 #12 P. 245 #2,12
P. 256 #6,7,11 P. 260 #5-7