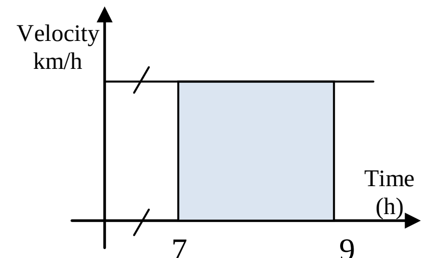
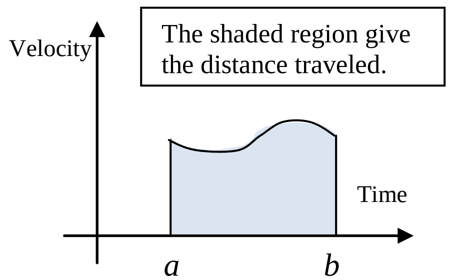
Estimating with Finite Sums

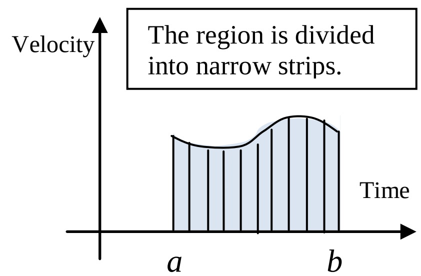
Date:

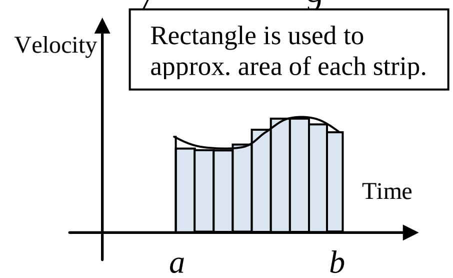
Areas as Limits



Story: A train moves along a track at a steady rate of 75 km/h from 7:00 am to 9:00am. What is the total distance traveled by the train?
Distance = Rate x Time , then Distance = 150 km.
Notice: Distance travelled is the area of the rectangle whose base is the time interval $[7,9]$ and whose height is the constant velocity 75.
What if the velocity is not steady?

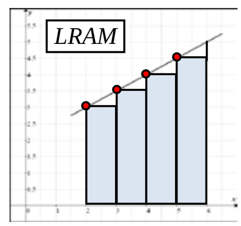


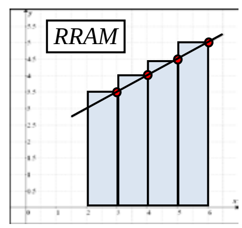


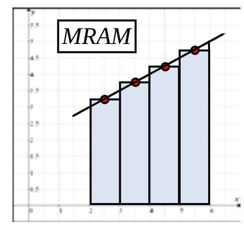


3 possible Rectangular Approximation Method (RAM)

LRAM: Left-hand endpoint Rectangular Approximation Method
RRAM: Right-hand endpoint Rectangular Approximation Method
MRAM: Midpoint Rectangular Approximation Method







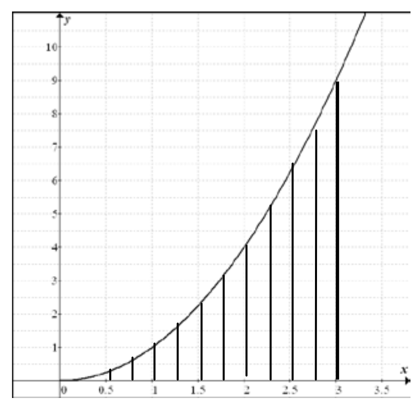
Example 1: Finding Distance Traveled when Velocity Varies

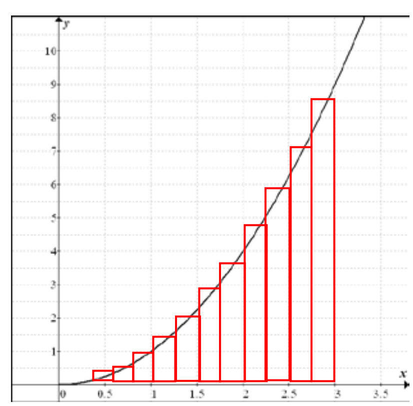
A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2$ for time $t \geq 0$. Where is the particle at $t = 3$? (Divide into _______ strips)

Divide into 12 thin strips, each with base $\Delta t = \dfrac{3}{12} = \dfrac{1}{4}$

By MRAM: Height = $f(m_i)$

Subinterval	$\left[0,\frac{1}{4}\right]$	$\left[\frac{1}{4},\frac{1}{2}\right]$	$\left[\frac{1}{2},\frac{3}{4}\right]$	$\left[\frac{3}{4},1\right]$	...	$\left[2\frac{3}{4},3\right]$
Midpoint m_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	...	$\frac{23}{8}$
Height = $(m_i)^2$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{25}{64}$	$\frac{49}{64}$	...	$\frac{529}{64}$
Area= $\frac{1}{4}(m_i)^2$	$\frac{1}{256}$	$\frac{9}{256}$	$\frac{25}{256}$	$\frac{49}{256}$	...	$\frac{529}{256}$





Total Approx Area :

$$= \frac{1+9+25+49+81+121+169+225+289+361+441+529}{256}$$

$$= \frac{2300}{256} \approx 8.98$$

Therefore the distance travelled is very close to 9 when $t = 3$.

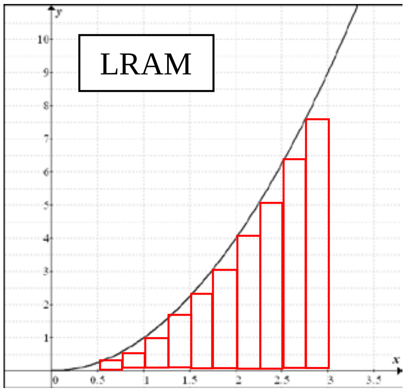
Example 2: LRAM vs RRAM

Repeat example 1 using LRAM and RRAM and compare the answers

Divide into 12 thin strips, each with base $\Delta t = \frac{3}{12} = \frac{1}{4}$

By LRAM: Height = $f(x_i)$

Subinterval	$\left[0,\frac{1}{4}\right]$	$\left[\frac{1}{4},\frac{1}{2}\right]$	$\left[\frac{1}{2},\frac{3}{4}\right]$	$\left[\frac{3}{4},1\right]$	...	$\left[2\frac{3}{4},3\right]$
Endpoint x_i Left	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	...	$2\frac{3}{4}$
Height = $(x_i)^2$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{9}{16}$	...	$\frac{121}{16}$
Area= $\frac{1}{4}(x_i)^2$	0	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{9}{64}$	...	$\frac{121}{64}$

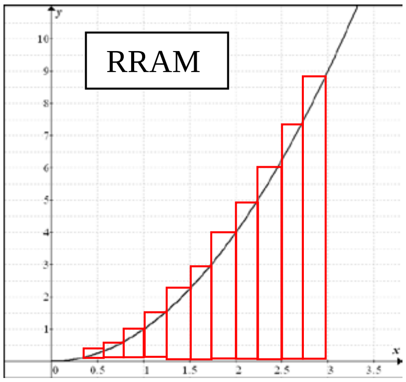


Total Approx. Area : (LRAM)

$$= \frac{0+1+4+9+16+25+36+49+64+81+100+121}{64} = \frac{506}{64} \approx 7.91$$

By RRAM: Height = $f(x_i)$

Subinterval	$\left[0,\frac{1}{4}\right]$	$\left[\frac{1}{4},\frac{1}{2}\right]$	$\left[\frac{1}{2},\frac{3}{4}\right]$	$\left[\frac{3}{4},1\right]$	...	$\left[2\frac{3}{4},3\right]$
Endpoint x_i Right	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	...	3
Height = $(x_i)^2$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{9}{16}$	1	...	9
Area= $\frac{1}{4}(x_i)^2$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{9}{64}$	$\frac{1}{4}$	...	$\frac{9}{4}$



Total Approx. Area : (RRAM)

$$= \frac{1+4+9+16+25+36+49+64+81+100+121+144}{64} = \frac{650}{64} \approx 10.16$$

Sigma Notation

$$\sum_{i=1}^{n} t_i = t_1 + t_2 + ... + t_n$$

Basic Properties of Sigma Notation

$$\sum_{i=1}^{n} c = nc \qquad \sum_{i=1}^{n} cf(x_i) = c\sum_{i=1}^{n} f(x_i) \qquad \sum_{i=1}^{n}(f_i \pm g_i) = \sum_{i=1}^{n} f_i \pm \sum_{i=1}^{n} g_i$$

Sum of a series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 3: Infinite Strips

Repeat Examples 1 & 2 with *n* strips

Divide into *n* thin strips, each with base $\Delta x = \frac{3}{n}$ Height = $f\left(\frac{3i}{n} - \frac{3}{n}\right) = f\left(\frac{3i-3}{n}\right)$ *LRAM*

Rectangle	1st	2nd	3rd		*i*th		*n*th
Subinterval	$\left[0, \frac{3}{n}\right]$	$\left[\frac{3}{n}, \frac{6}{n}\right]$	$\left[\frac{6}{n}, \frac{9}{n}\right]$	...	$\left[\frac{3i-3}{n}, \frac{3i}{n}\right]$	...	$\left[3 - \frac{3}{n}, 3\right]$
Endpoint x_i Left	0	$\frac{3}{n}$	$\frac{6}{n}$	...	$\frac{3i-3}{n}$	...	$3 - \frac{3}{n}$
Height = $(x_i)^2$	0	$\left(\frac{3}{n}\right)^2$	$\left(\frac{6}{n}\right)^2$	...	$\left(\frac{3i-3}{n}\right)^2$	...	$\left(3 - \frac{3}{n}\right)^2$
Area= $\frac{3}{n}(x_i)^2$	0	$\frac{3}{n}\left(\frac{3}{n}\right)^2$	$\frac{3}{n}\left(\frac{6}{n}\right)^2$	...	$\frac{3}{n}\left(\frac{3i-3}{n}\right)^2$	...	$\frac{3}{n}\left(3 - \frac{3}{n}\right)^2$

Total Approx. Area : (LRAM)

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} f\left(\frac{3i-3}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\frac{3i-3}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2 - 18i + 9}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n^3}\left[\sum_{i=1}^{n} 9i^2 - \sum_{i=1}^{n}(18i) + \sum_{i=1}^{n} 9\right]$$

$$= \lim_{n\to\infty} \frac{3}{n^3}\left[9\sum_{i=1}^{n} i^2 - 18\sum_{i=1}^{n} i + 9\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + 1\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{2(n)(n+1)}{2} + 1n\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{n(n+1)(2n+1) - 6n(n+1) + 6n}{6}\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left[\frac{2n^3 + \ldots\ldots}{6}\right]$$

$$= \lim_{n\to\infty} \frac{9n^3 + \ldots}{n^3} = 9$$

Area = 9 (LRAM)

Divide into *n* thin strips, each with base $\Delta x = \frac{3}{n}$ Height = $f\left(\frac{3i}{n}\right)$ *RRAM*

Rectangle	1st	2nd	3rd		*i*th		*n*th
Subinterval	$\left[0, \frac{3}{n}\right]$	$\left[\frac{3}{n}, \frac{6}{n}\right]$	$\left[\frac{6}{n}, \frac{9}{n}\right]$	...	$\left[\frac{3i-3}{n}, \frac{3i}{n}\right]$	...	$\left[3 - \frac{3}{n}, 3\right]$
Endpoint x_i Right	$\frac{3}{n}$	$\frac{6}{n}$	$\frac{9}{n}$	...	$\frac{3i}{n}$	...	3
Height = $(x_i)^2$	$\left(\frac{3}{n}\right)^2$	$\left(\frac{6}{n}\right)^2$	$\left(\frac{9}{n}\right)^2$	...	$\left(\frac{3i}{n}\right)^2$	...	$(3)^2$
Area= $\frac{3}{n}(x_i)^2$	$\frac{3}{n}\left(\frac{3}{n}\right)^2$	$\frac{3}{n}\left(\frac{6}{n}\right)^2$	$\frac{3}{n}\left(\frac{9}{n}\right)^2$	...	$\frac{3}{n}\left(\frac{3i}{n}\right)^2$	...	$\frac{3}{n}(3)^2$

Estimating with Finite Sums Date:

Total Approx. Area : (RRAM)

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}f\left(\frac{3i}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\frac{3i}{n}\right)^2$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}\right) = \lim_{n\to\infty}\frac{3}{n^3}\left[\sum_{i=1}^{n}9i^2\right]$$

$$= \lim_{n\to\infty}\frac{3}{n^3}\left[9\sum_{i=1}^{n}i^2\right] = \lim_{n\to\infty}\frac{27}{n^3}\left[\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\left[\frac{2n^3+......}{6}\right]$$

$$= \lim_{n\to\infty}\frac{9n^3+...}{n^3} = 9$$

Area = 9 (RRAM)

Divide into n thin strips, each with base $\Delta x = \frac{3}{n}$ Height = $f\left(\frac{6i-3}{2n}\right)$ *MRAM*

Rectangle	1st	2nd	3rd		ith		nth
Subinterval	$\left[0,\frac{3}{n}\right]$	$\left[\frac{3}{n},\frac{6}{n}\right]$	$\left[\frac{6}{n},\frac{9}{n}\right]$	...	$\left[\frac{3i-3}{n},\frac{3i}{n}\right]$	...	$\left[3-\frac{3}{n},3\right]$
Midpoint m_i	$\frac{3}{2n}$	$\frac{9}{2n}$	$\frac{15}{2n}$	...	$\frac{6i-3}{2n}$	...	$\frac{6n-3}{2n}$
Height = $(m_i)^2$	$\left(\frac{3}{2n}\right)^2$	$\left(\frac{9}{2n}\right)^2$	$\left(\frac{15}{2n}\right)^2$	...	$\left(\frac{6i-3}{2n}\right)^2$	...	$\left(\frac{6n-3}{2n}\right)^2$
Area= $\frac{3}{n}(m_i)^2$	$\frac{3}{n}\left(\frac{3}{2n}\right)^2$	$\frac{3}{n}\left(\frac{9}{2n}\right)^2$	$\frac{3}{n}\left(\frac{15}{2n}\right)^2$	...	$\frac{3}{n}\left(\frac{6i-3}{2n}\right)^2$	...	$\frac{3}{n}\left(\frac{6n-3}{2n}\right)^2$

Total Approx. Area : (MRAM)

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}f\left(\frac{6i-3}{2n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left(\frac{6i-3}{2n}\right)^2$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{36i^2-36i+9}{4n^2}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n^3}\left[\sum_{i=1}^{n}9i^2-\sum_{i=1}^{n}(9i)+\sum_{i=1}^{n}\frac{9}{4}\right]$$

$$= \lim_{n\to\infty}\frac{3}{n^3}\left[9\sum_{i=1}^{n}i^2-9\sum_{i=1}^{n}i+\frac{9}{4}\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\left[\sum_{i=1}^{n}i^2-\sum_{i=1}^{n}i+\frac{1}{4}\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}-\frac{(n)(n+1)}{2}+\frac{n}{4}\right]$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\left[\frac{2n(n+1)(2n+1)-6n(n+1)+3n}{12}\right]$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\left[\frac{4n^3+......}{12}\right]$$

$$= \lim_{n\to\infty}\frac{9n^3+...}{n^3} = 9$$

Area = 9 (MRAM)

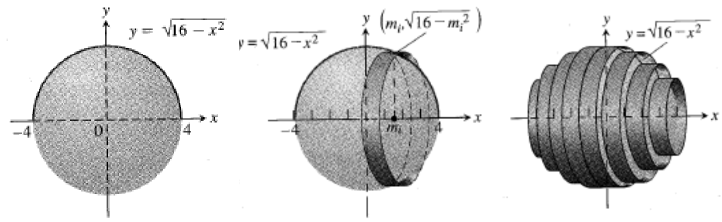
$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x \quad \text{where } \Delta x = \frac{b-a}{n} \text{ and } x_i = a+i\Delta x$$

$$\text{LRAM} = \text{MRAM} = \text{RRAM as } n\to\infty$$

Example 4: Volume of Sphere

Estimate the volume of a solid sphere of radius 4.

Using Formula: $Volume = \frac{4}{3}\pi(4)^3 = \frac{256\pi}{3}$



Divide into n thin strips, each with base

$\Delta x = \frac{4-(-4)}{n} = \frac{8}{n}$ Height = $f\left(\frac{6i-3}{2n}\right)$ *MRAM*

Rectangle	1st	2nd		ith
Subinterval	$\left[-4, -4+\frac{8}{n}\right]$	$\left[-4+\frac{8}{n}, -4+\frac{16}{n}\right]$	...	$\left[-4+\frac{8i-8}{n}, -4+\frac{8i}{n}\right]$
Midpoint m_i	$-4+\frac{4}{n}$	$-4+\frac{12}{n}$	...	$-4+\frac{8i-4}{n}$
Radius = $\sqrt{16-(m_i)^2}$	$\sqrt{16-\left(-4+\frac{4}{n}\right)^2}$	$\sqrt{16-\left(-4+\frac{12}{n}\right)^2}$	...	$\sqrt{16-\left(-4+\frac{8i-4}{n}\right)^2}$
$Volume = \pi r^2 h$ $= \pi\left[\sqrt{16-(m_i)^2}\right]^2 \frac{8}{n}$ $= \frac{8\pi}{n}\left(16-(m_i)^2\right)$	$\frac{8\pi}{n}\left[16-\left(-4+\frac{4}{n}\right)^2\right]$	$\frac{8\pi}{n}\left[16-\left(-4+\frac{12}{n}\right)^2\right]$	...	$\frac{8\pi}{n}\left[16-\left(-4+\frac{8i-4}{n}\right)^2\right]$

$$V = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{8\pi}{n}\left[16-\left(-4+\frac{8i-4}{n}\right)^2\right]$$

$$= \lim_{n\to\infty}\frac{8\pi}{n}\sum_{i=1}^{n}\left[16-\left(16-8\left(\frac{8i-4}{n}\right)+\left(\frac{8i-4}{n}\right)^2\right)\right]$$

$$= \lim_{n\to\infty}\frac{8\pi}{n}\sum_{i=1}^{n}\left[8\left(\frac{8i-4}{n}\right)-\left(\frac{8i-4}{n}\right)^2\right]$$

$$= \lim_{n\to\infty}\frac{8\pi}{n}\sum_{i=1}^{n}\frac{8n(8i-4)-(8i-4)^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{8\pi}{n^3}\sum_{i=1}^{n}\left[8n(8i-4)-\left(64i^2-32i+16\right)\right]$$

$$= \lim_{n\to\infty}\frac{8\pi}{n^3}\left[\sum_{i=1}^{n}32n(2i-1)-16\left(4i^2-4i+1\right)\right]$$

$$= \lim_{n\to\infty}\frac{128\pi}{n^3}\sum_{i=1}^{n}\left[2n(2i-1)-4i^2+4i-1\right]$$

$$= \lim_{n\to\infty}\frac{128\pi}{n^3}\left[2n\sum_{i=1}^{n}(2i-1)-4\sum_{i=1}^{n}i^2+4\sum_{i=1}^{n}i-\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\frac{128\pi}{n^3}\left[4n\sum_{i=1}^{n}i-2n\sum_{i=1}^{n}1-4\sum_{i=1}^{n}i^2+4\sum_{i=1}^{n}i-\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\frac{128\pi}{n^3}\left[4n\frac{(n)(n+1)}{2}-2n(n)-4\frac{n(n+1)(2n+1)}{6}+4\frac{(n)(n+1)}{2}-n\right]$$

$$= \lim_{n\to\infty}\frac{128\pi}{n^3}\left[2n(n)(n+1)-2n^2-\frac{2}{3}n(n+1)(2n+1)+2(n)(n+1)-n\right]$$

$$= \lim_{n\to\infty}\frac{128\pi}{n^3}\left(2n^3+\ldots-2n^2-\frac{4}{3}n^3+\ldots+2n^2-n\right)$$

$$= \lim_{n\to\infty}\frac{128\pi}{n^3}\left(\frac{2}{3}n^3+\ldots\right)$$

$$= \lim_{n\to\infty}\left(\frac{256\pi n^3}{3n^3}+\ldots\right)$$

$$= \frac{256\pi}{3}$$

Homework:
P. 270 # 1-4, 9-12,17, 20

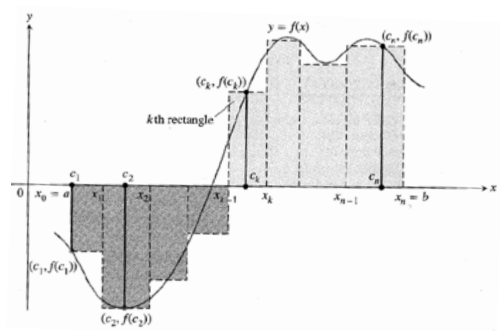
Definite Integrals

Date:

- **Riemann Sums**
- For Rectangular Approximation Method, we approximate the area under the graph by rectangles of equal width. The heights of the rectangles are the values of $f(x)$ at the endpoints or midpoints of the subintervals. In Riemann Sum approximations, we relax these requirements; the rectangles **need not have equal width** and the height of a rectangle may be ***any*** value of $f(x)$ within the subinterval. More formally, to define a Riemann sum, we choose a partition and a set of intermediate points. A **partition** *P* of length *N* is any choice of points in $[a,b]$ that divides the interval into *N* subintervals:

Georg Friedrich Riemann (1826-1866)
One of the greatest mathematicians of the 19th century and perhaps second only to his teacher C.F. Gauss.

- Partition *P:* $a = x_0 < x_1 < x_2 .... < x_N = b$
- Subintervals: $[x_0, x_1], [x_1, x_2], ... [x_{N-1}, x_N]$
- We denote the length of the i^{th} interval by $\Delta_i = x_i - x_{i-1}$
- Begins with a continuous function $y = f(x)$ defined on a closed interval $[a,b]$.
- Partition the interval into *n* subintervals by choosing *n* – 1 points (not necessary regular interval) start from $x_1, x_2, ..., x_{n-1}$ with $a = x_0$ and $b = x_n$. (**Partition** of $[a,b]$)
- The length between x_0 and x_1 (the 1st subinterval) is Δx_1, so the length of the k^{th} subinterval is Δx_k.

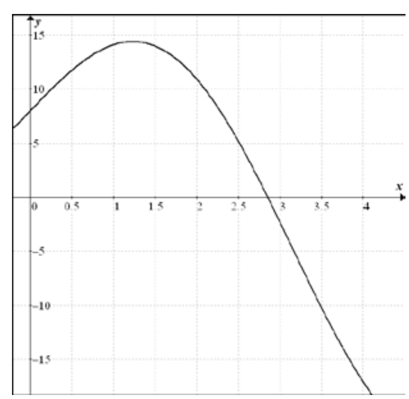


- Select a number in each subinterval and connect it to the curve from the *x*-axis. Denote the number chosen from the k^{th} subinterval is c_k. So the point on the curve is $(c_k, f(c_k))$.
- Stand a vertical rectangle that touches the *x*-axis and the curve at $(c_k, f(c_k))$. The area of such rectangle should be $f(c_k) \bullet \Delta x_k$
- The sum of all the rectangles in positive or negative areas is a

  Riemann sum for *f* on the interval $[a,b]$ ➔ $s_n = \sum_{k=1}^{n} f(c_k) \bullet \Delta x_k$
- The maximum of the lengths Δ_{xi} is called the **norm** of the partition *P* and is denoted $\|P\|$

Example 1: Riemann Sum

Let $f(x) = 8 + 12\sin x - 4x$ Calculate the Riemann sum for the partition *P* of $[0,4]$



A function is **integrable** over $[a,b]$ if all of the Riemann sums approach the same limit ***L*** as the norm of the partition tends to zero. More formally, we write:

Definite Integral
Let *f* be a continuous function defined on an interval $[a,b]$.
The **definite integral** of *f* from *a* to *b* is:

$$L=\int_a^b f(x)dx=\lim_{\|P\|\to 0} Riemann\ Sum=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where } \Delta x=\frac{b-a}{n} \text{ and } x_i=a+i\Delta x$$

The symbol $\int$ is called an **integral** sign which stands for a limit of sums. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and *a* and *b* are called the **limits of integration**; *a* is the **lower limit** and *b* is the **upper limit**. The procedure of calculating an integral is called **integration**.

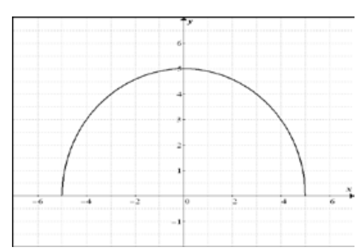
Example 2: Using the Notation

The interval $[-1,3]$ is partitioned into *n* subintervals of equal length $\Delta x=\frac{4}{n}$. Let m_k denote the midpoint of the k^{th} subinterval. Express the limit $\lim_{n\to\infty}\sum_{k=1}^{n}\left[3(m_k)^2-2m_k+5\right]\Delta x$ as an integral.

Area Under a Curve (as a Definite Integral)
The area under the curve $y=f(x)$ from *a* to *b* is the integral of *f* from *a* to *b* ➔ $A=\int_a^b f(x)dx$

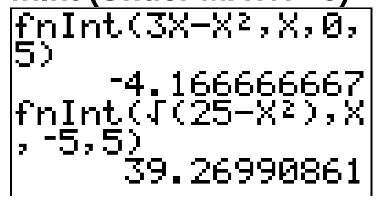
Example 3: Area under a Curve (Before knowing Antiderivative)

Evaluate the integral $\int_{-5}^{5}\sqrt{25-x^2}\,dx$



Example 4: Area under a Curve (Before knowing Antiderivative)

Using Ti-Calculator
fnInt (Under MATH - 9)



fnInt(Expression,Variable,Lower,Upper)

Evaluate the integral $\int_0^5 \left(3x - x^2\right)dx$

$f(x) = 3x - x^2$, lower limit a = 0, and upper limit b = 5.

$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$ $\qquad x_i = a + i\Delta x = 0 + i\left(\frac{5}{n}\right) = \frac{5i}{n}$

$$\int_0^5 \left(3x - x^2\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\left(\frac{5}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(\frac{5i}{n}\right) - \left(\frac{5i}{n}\right)^2\right]\frac{5}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{75i}{n^2} - \frac{125i^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{75}{n^2}\sum_{i=1}^{n} i - \frac{125}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{75}{n^2}\right)\frac{n(n+1)}{2} - \left(\frac{125}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right]$$

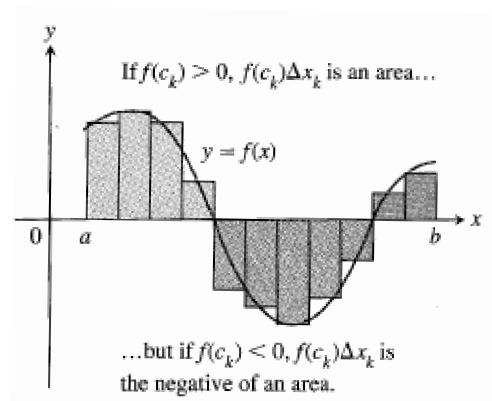
$$= \lim_{n\to\infty}\left[\frac{75}{2}\left(1 + \frac{1}{n}\right) - \frac{125}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right]$$

$$= \frac{75}{2}(1) - \frac{125}{6}(1)(2)$$

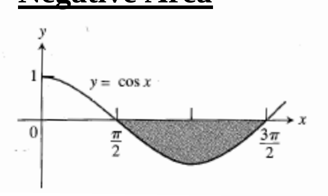
$$= \frac{-25}{6}$$

Method 2: After knowing antiderivatives

$\int_0^5 \left(3x - x^2\right)dx$



Negative Area



$\because \int_a^b f(x)dx = -(\text{the area})\text{ if } f(x) \le 0$

$\therefore \text{Area} = -\int_a^b f(x)dx \text{ when } f(x) \le 0$

$\int_a^b f(x)dx = (\text{area above the } x\text{-axis}) - (\text{area below the } x - \text{axis})$

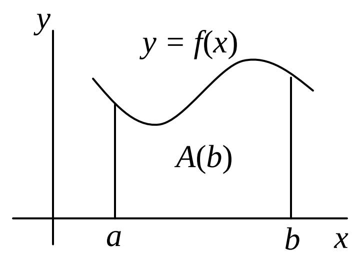
Homework:
P. 282 #1-40

Definite and Indefinite Integrals (Antiderivatives)

Date

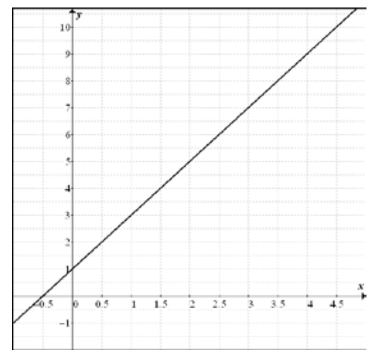
Area under a Curve

If $y = f(x)$ is a positive function, the **area of the region under** $y = f(x)$ from ***a*** to ***b*** is the area of the region below $y = f(x)$ and above the *x*-axis (y = 0), to the right of the vertical line $x = a$ and to the left of $x = b$. Such region is denoted as $A(b)$.



Example 1: Area of a Trapezoid

Sketch the region under $y = 2x + 1$ from 1 to *x*. Find the area function.



In general:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2} h\left[f(x) + f(x+h)\right]}{h}$$

$$= \lim_{h \to 0} \frac{f(x) + f(x+h)}{2}$$

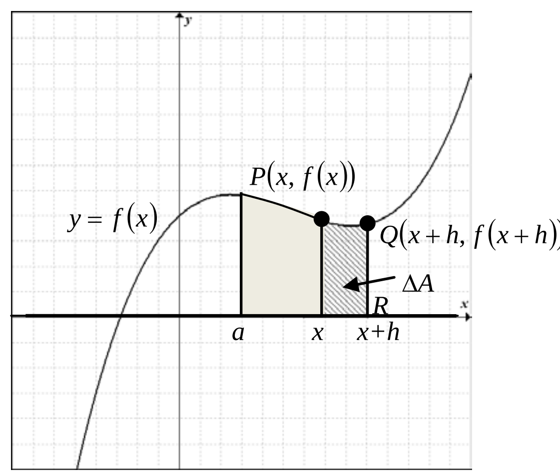
$$= \frac{f(x) + f(x)}{2} = \frac{2f(x)}{2} = f(x)$$

$$\therefore A'(x) = f(x)$$

Area of the Trapezoid

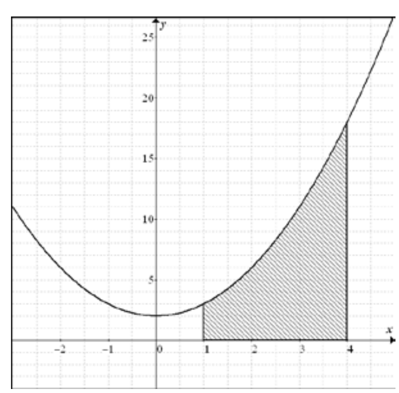
$$A(x+h) - A(x)$$

$$\approx \frac{1}{2} h\left[f(x) + f(x+h)\right]$$



Example 2: Apply the rule

Find the area under $y = x^2 + 2$ from $x = 1$ to $x = 4$



Ingeneral

$A'(x) = f(x)$

$A(x) = F(x) + C$

$A(a) = 0$ (Area of a line is 0)

$0 = F(a) + C$

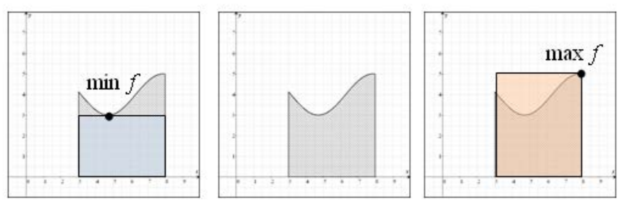
$C = -F(a)$

$A(x) = F(x) - F(a)$

$A(b) = F(b) - F(a)$

$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$

Rules for Definite Integrals

Order of Integration	$\int_b^a f(x)dx = -\int_a^b f(x)dx$
Zero	$\int_a^a f(x)dx = 0$
Constant Multiple	$\int_a^b kf(x)dx = k\int_a^b f(x)dx$
Sum and Difference	$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
Additivity	$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
Inequality (See diagram)	$\min f \cdot (b-a) \le \int_a^b f(x)dx \le \max f \cdot (b-a)$ max *f* and min *f* are the maximum and minimum values of *f* on [*a*, *b*] 
Domination	$f(x) \ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)dx \ge \int_a^b g(x)dx$

Example 3: Using the Rules for Definite Integrals

Suppose $\int_{-1}^{1} f(x)dx = 5$, $\int_{1}^{4} f(x)dx = -2$, and $\int_{-1}^{1} \mathrm{h}(\mathrm{x})dx = 7$

Find each of the following integrals, if possible.

a) $\int_4^1 f(x)dx$

b) $\int_{-1}^4 f(x)dx$

c) $\int_{-1}^1 [2f(x) + 3h(x)]dx$

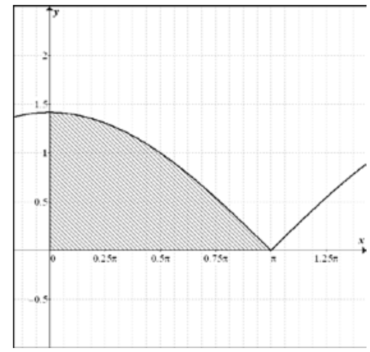
d) $\int_0^1 f(x)dx$

e) $\int_{-2}^2 h(x)dx$

f) $\int_{-1}^4 [f(x) + h(x)]dx$

Example 4: Finding bounds for an Integral

Find the bounds of $\int_0^{\pi} \sqrt{1+\cos x}\,dx$.



Average (Mean) Value on [*a*, *b*]

Recall $\Delta x = \frac{b-a}{n} \Rightarrow \frac{1}{n} = \frac{\Delta x}{b-a}$

and $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

Average (Mean) Value on [*a*, *b*]

$$\text{average}(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

The average of the n sampled values is:

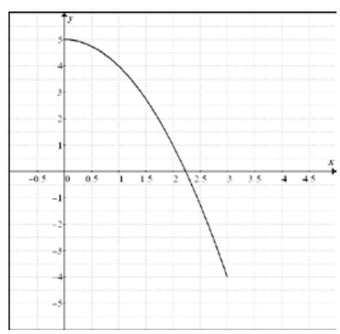
$$\text{average } f = \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n} \quad \text{assume } n \to \infty$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{\Delta x}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)$$

$$= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 5: Average Value of the function

Find the average value of $f(x) = 5 - x^2$ on $[0,3]$. Does f actually take on this value at some point in the given interval?



Using Ti-Calculator
fnInt (Under MATH - 9)

```
fnInt(5-X²,X,0,3
)
                6
```

fnInt(Expression,Variable,Lower,Upper)

Indefinite Integrals - Antiderivatives

If F is an antiderivative of f on an interval where $F'(x) = f(x)$, then the most general antiderivative of f on that interval is $F(x)+C$ where C is an arbitrary constant.

The antiderivative $\int f(x)dx$ is often called an indefinite integral of f. $\int f(x)dx = F(x)+C$

Definite and Indefinite Integrals (Antiderivatives) Date

Table of Indefinite Integrals (Antiderivatives)		
$\int 0dx = C$	$\int Cdx = Cx + C_2$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$
$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$	$\int a^{kx}dx = \frac{a^{kx}}{k\ln a} + C$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\int \sin kxdx = \frac{-1}{k}\cos kx + C$	$\int \cos kxdx = \frac{1}{k}\sin kx + C$	$\int \sec^2 kxdx = \frac{1}{k}\tan kx + C$
$\int \csc kx\cot kxdx = \frac{-1}{k}\csc kx + C$	$\int \sec kx\tan kxdx = \frac{1}{k}\sec kx + C$	$\int \csc^2 kxdx = \frac{-1}{k}\cot kx + C$
$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + C$	$\int \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1}x + C$	$\int \frac{1}{x^2+1}dx = \tan^{-1}x + C$

<u>Example 6: Antiderivatives (Indefinite Integrals)</u>

Find the antiderivative of *f*.

a) $f(x) = 2x^2 - x + 7$

b) $f(x) = \cos - \sin x$

c) $\int \left(-3e^{-x} + 6e^{2x}\right)dx$

d) $\int \left(\frac{2}{x^2} - \frac{5}{x} + x\right)dx$

e) $\int \left(6x^2 + \csc^2 x\right)dx$

<u>Fundamental Theorem of Calculus – Integral Evaluation Theorem</u>

If f is continuous at every point of $[a,b]$, and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$ (To be discussed later in detail)

<u>Example 7: Integral Evaluation Theorem</u>

Find $\int_{\pi}^{2\pi} \sin xdx$

Homework:
P. 290 #1-7, 13-15, 19-35

Fundamental Theorem of Calculus

Date

Fundamental Theorem 1
If f is continuous on $[a, b]$, then the function
$F(x)=\int_a^x f(t)dt$ has a derivative at every point x in $[a, b]$,
and $\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)dt=f(x) \qquad a\in C$

Proof

$$\frac{d}{dx}\int_a^x f(t)dt=\frac{d}{dx}F(t)\Big]_a^x$$
$$=\frac{d}{dx}\left[F(x)-F(a)\right]=f(x)$$

Example 1: Applying the Fundamental Theorem 1

Find a) $\frac{d}{dx}\int_2^x 3t^2dt$ when $x=4$ b) $\frac{d}{dx}\int_{-\pi}^x \cos t dt$ c) $\frac{d}{dx}\int_0^x \frac{1}{1+t^2}dt$

Example 2: Applying the Fundamental Theorem 1 with the Chain Rule

Find $\frac{dy}{dx}$ if $y=\int_1^{x^3}\tan t dt$

$$\frac{d}{dx}\int_a^u f(t)dt=f(u)\cdot\frac{du}{dx}$$

Recall:
i) $\int_a^b f(x)dx=-\int_b^a f(x)dx$
ii) $\int_a^b f(x)dx=\int_0^b f(x)dx-\int_0^a f(x)dx$

Example 3: Variable Lower Limits of Integration

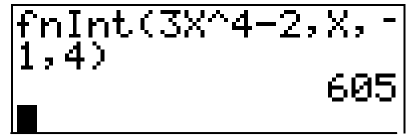
Find $\frac{dy}{dx}$ if

a) $y=\int_x^5 3t\sin t dt$ b) $y=\int_{2x}^{x^2}\frac{1}{2+e^t}dt$

Fundamental Theorem 2 – Integral Evaluation Theorem
If f is continuous at every point of $[a, b]$, and if F is any antiderivative of f on $[a, b]$, then
$$\int_a^b f(x)dx=F(x)\Big]_a^b=F(b)-F(a)$$

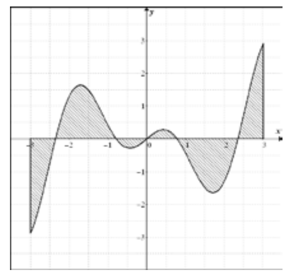
Example 4: Evaluating an Integral

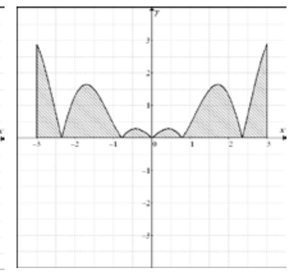
Evaluate $\int_{-1}^{4}\left(3x^4-2\right)dx$



Example 5: Find Area using fnInt

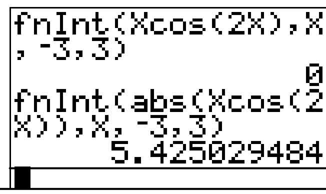
Find the area of the region between the curve $y = x\cos 2x$ and the *x*-axis over the interval $-3 \le x \le 3$.





Not until we learn "Substitution Rule", we cannot perform the antiderivative for $y = x\cos 2x$, but we can still use TI Calculator to find the area.

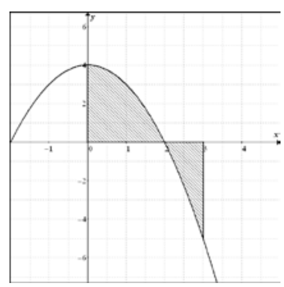
$\therefore \int_{-3}^{3} x\cos 2x dx = 5.43$

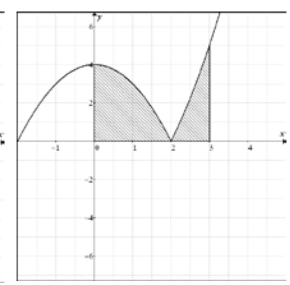


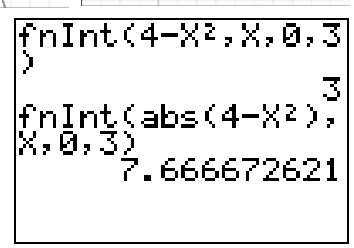
To find area between the graph of $y = f(x)$ and the *x*-axis over the interval $[a,b]$ numerically using TI-83 ➔ $\int_a^b f(x)dx = fnInt(|f(x)|, x, a, b)$

Example 6: Find Area using Antiderivatives

Find the area of the region between the curve $y = 4 - x^2$ and the *x*-axis over the interval $0 \le x \le 3$.







<u>Example 7: Using the Graph of *f* to analyze antiderivatives</u>

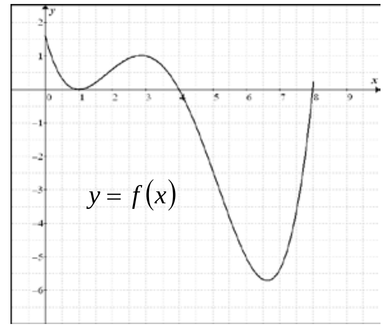
The graph of a continuous function *f* with domain [0,8] is shown in the diagram.

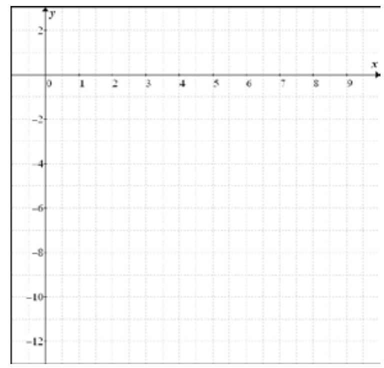
Let *h* be the function defined by $h(x)=\int_1^x f(t)dt$.

$h'(x)=f(x)$

$h(x)=\int_a^x f(t)dt$

a) Find $h(1)$
b) Is $h(0)$ positive or negative?
c) Find the value of x for which $h(x)$ is a maximum & minimum.
d) Find the coordinates of all points of inflections of the graph of $y=h(x)$.



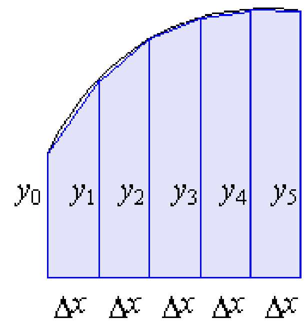


<u>Homework:</u>
P. 302 #1-50

Trapezoidal Rule & Simpson's Rule

Date

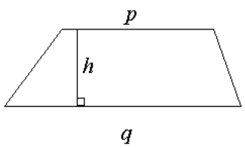
Trapezoidal Rule



Trapezoidal rule (**trapezoid rule** or **trapezium rule**) is an approximate technique for calculating the definite integral. $\int_a^b f(x)\,dx.$

Instead of using rectangles, (Rectangular Approximation Method – RAM) we see that **trapezoids** (trapeziums) give a better approximation to the area.

Recall: Area of a trapezoid (trapezium) is given by:



$$\text{Area} = \frac{h}{2}(p+q) \qquad h = \Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx \approx (b-a)\frac{f(a)+f(b)}{2}.$$

The trapezoidal rule works by approximating the region under the graph of the function $f(x)$ as a trapezoid and calculating its area.

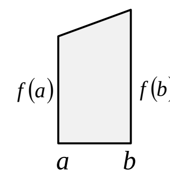
$$Area \approx \frac{1}{2}(y_0+y_1)\Delta x + \frac{1}{2}(y_1+y_2)\Delta x + \frac{1}{2}(y_2+y_3)\Delta x \ldots$$

$$\approx \Delta x\left(\frac{y_0}{2}+y_1+y_2+y_3+\ldots+\frac{y_n}{2}\right)$$

$$\approx \frac{\Delta x}{2}(y_0+2y_1+2y_2+\ldots 2y_{n-1}+y_n)$$

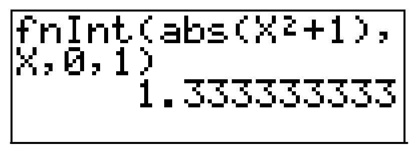
for n trapezoids where

$y_0 = f(a), y_1 = f(x_1), \ldots, y_n = f(b)$



Example 1: Applying the Trapezoidal Rule

Use the Trapezoidal rule with n = 5, approximate $\int_0^1 (x^2+1)dx$



Example 2: Application to Trapezoidal Rule

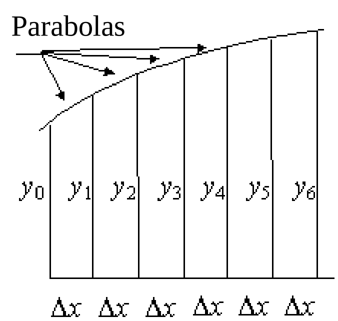
An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in Celsius in the following table. What was the average temperature for the 12-hour period?

Time	N	1	2	3	4	5	6	7	8	9	10	11	M
Temp	17	18	19	20	21	20	20	18	18	17	16	14	13

Recall:
Average (Mean) Value on [*a*, *b*]

$$\text{average}(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Simpson's Rule



The Trapezoidal Rule is an improvement over using rectangles because we have much less "missing" from our calculations. We used **straight lines** to model the curve in trapezoidal Rule.

We seek an even better approximation. In **Simpson's Rule**, we use **parabolas** to approximate each part of the curve. This proves to be very efficient. (i.e., parabolic arcs instead of the straight line segments used in the trapezoidal rule).

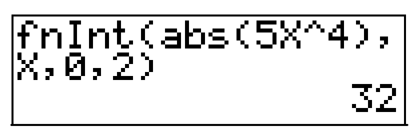
We can show (by integrating the area under each parabola and adding these areas) that the approximate area is given by **Simpson's Rule.**

$$Area = \int_a^b f(x)dx \approx \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

where $[a,b]$ is partitioned into an *even* number n of subintervals of equal length $\Delta x = \frac{b-a}{n}$.

Example 3: Applying the Simpson's Rule

Use the Simpson's rule with n = 4, approximate $\int_0^2 5x^4 dx$

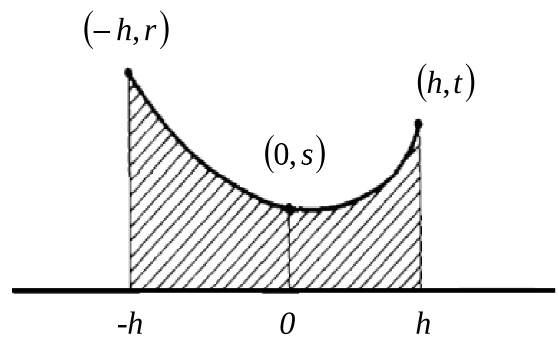


Homework:
P. 312 #2-18, 23-26

Proof of Simpson's Rule

The proof is easier to understand if the *y*-axis is drawn through the second point, so that $u+h=0$, and the three points have coordinates $(-h,r),(0,s),(h,t)$

Suppose the parabola has the equation $y=ax^2+bx+c$. Then the area under the parabola is:



$$A=\int_{-h}^{h}\left(ax^2+bx+c\right)dx=\left.\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right]_{-h}^{h}$$

$$=\left(\frac{ah^3}{3}+\frac{bh^2}{2}+ch\right)-\left(\frac{-ah^3}{3}+\frac{bh^2}{2}-ch\right)=\frac{2}{3}ah^3+2ch$$

Substitute the three points $(-h,r),(0,s),(h,t)$ into the parabola $y=ax^2+bx+c$

$r=ah^2-bh+c; \quad s=c; \quad t=ah^2+bh+c$

$r+t=2ah^2+2c$

$a=\dfrac{r+t-2c}{2h^2}$

Sub *a* into *A*

$$A=\frac{2}{3}ah^3+2ch$$

$$=\frac{2}{3}\left(\frac{r+t-2c}{2h^2}\right)h^3+2ch$$

$$=\frac{(r+t-2c)h}{3}+2ch=\frac{h(r+t-2c)+6ch}{3}$$

$$=\frac{h}{3}(r+t-2c+6c)=\frac{h}{3}(r+4c+t)$$

$A=\dfrac{\Delta x}{3}\left[f(x_k)+4f(x_{k+1})+f(x_{k+2})\right]$ is the area of the region under **one** parabolic arc from x_k to x_{k+2}

$$A=\sum_a^b\frac{\Delta x}{3}\left[f(x)+4f(x+\Delta x)+f(x+2\Delta x)\right]$$

$$=\frac{\Delta x}{3}\left\{\left[f(x_0)+4f(x_1)+f(x_2)\right]+\left[f(x_2)+4f(x_3)+f(x_4)\right]+\ldots\right\}$$

$$=\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(f_2)+4f(x_3)+2f(x_4)+\ldots.+4f(x_{n-1})+f(x_n)\right]$$

In general:

For a continuous function *f* on [*a*, *b*],

$$Area=\int_a^b f(x)dx\approx\frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\ldots+2y_{n-2}+4y_{n-1}+y_n\right)$$

where $[a,b]$ is partitioned into an *even* number *n* of subintervals of equal length $\Delta x=\dfrac{b-a}{n}$.