Proving the Cross Product

To develop a formula for $\vec{a} \times \vec{b}$

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ and let $\vec{v} = (x, y, z)$ be the vector that is perpendicular to $\vec{a}$ and $\vec{b}$.

So: (1) $\vec{a} \cdot \vec{v} = (a_1, a_2, a_3) \cdot (x, y, z) = a_1 x + a_2 y + a_3 z = 0$

(2) $\vec{b} \cdot \vec{v} = (b_1, b_2, b_3) \cdot (x, y, z) = b_1 x + b_2 y + b_3 z = 0$

A system of two equations in three unknowns, which has an infinite number of solutions. To solve this system of equations, we will multiply the first equation by b_1 and the second equation by a_1 and then subtract.

(1) $\times b_1 \rightarrow$ (3) $b_1 a_1 x + b_1 a_2 y + b_1 a_3 z = 0$

(2) $\times a_1 \rightarrow$ (4) $a_1 b_1 x + a_1 b_2 y + a_1 b_3 z = 0$

(3) – (4)

$(b_1 a_2 - a_1 b_2) y = (a_1 b_3 - b_1 a_3) z$

Multiply each side by -1 and rearranging gives the desired result:

$(a_1 b_2 - b_1 a_2) y = (b_1 a_3 - a_1 b_3) z$

$$\Rightarrow \quad \frac{y}{a_3 b_1 - a_1 b_3} = \frac{z}{a_1 b_2 - a_2 b_1}$$

If we carry out an identical procedure and eliminate *z* from the system of equations, we have the following:

$$\frac{x}{a_2 b_3 - a_3 b_2} = \frac{y}{a_3 b_1 - a_1 b_3}$$

If we combine the two statements and set them equal to a constant *k*, we have

$$\frac{x}{a_2 b_3 - a_3 b_2} = \frac{y}{a_3 b_1 - a_1 b_3} = \frac{z}{a_1 b_2 - a_2 b_1} = k$$

Therefore $k(a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$, $k \in R$

If $k = 1$, then $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

If $k = -1$, then $\vec{b} \times \vec{a} = (a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_2 b_1 - a_1 b_2)$