Proving the Product Rule, Quotient Rule & Chain Rule

Date:

POWER RULE

If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$

In Leibniz notation, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \text{ where } f(x) = x^n$$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{(x+h-x)\left[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}\right]}{h}$$

$$= \lim_{h\to 0}\left[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}\right]$$

$$= x^{n-1} + x^{n-2}(x) + \ldots + (x)x^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + \ldots + x^{n-1} + x^{n-1} \qquad \text{(Since there are } n \text{ terms)}$$

$$= nx^{n-1}$$

Factor $(x+h)^n - x^n$

$(x+h) - x$ is the root

$$\begin{array}{r} (x+h)^{n-1} + (x+h)^{n-2}x + \ldots (x+h)^0 x^{n-1}. \\ (x+h)-x\,\overline{\big)\,(x+h)^n + \ldots\ldots\ldots\ldots\ldots\ldots - x^n} \\ \underline{(x+h)^n - x(x+h)^{n-1}} \\ x(x+h)^{n-1} \\ \underline{x(x+h)^{n-1} - x^2(x+h)^{n-2}} \\ x^2(x+h)^{n-2} \\ \ldots\text{ etc} \end{array}$$

PRODUCT RULE

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$.

If u and v are functions of x, $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

$$\text{Recall: } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$p(x) = f(x)g(x)$$

$$p'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left\{\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h) + f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0}g(x+h) + \lim_{h\to 0}f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= f'(x)g(x) + f(x)g'(x)$$

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QUOTIENT RULE

If $p(x)=\frac{f(x)}{g(x)}$, then $p'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$.

$p(x)=\frac{f(x)}{g(x)}$, ➔ $f(x)=p(x)g(x)$

$f'(x)=p'(x)g(x)+p(x)g'(x)$ (Product Rule)

$p'(x)g(x)=f'(x)-p(x)g'(x)$

$p'(x)=\frac{f'(x)-p(x)g'(x)}{g(x)}$

$p'(x)=\frac{f'(x)-\frac{f(x)}{g(x)}g'(x)}{g(x)}$ Recall: $p(x)=\frac{f(x)}{g(x)}$

$p'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ (Multiply $\frac{g(x)}{g(x)}$)

CHAIN RULE

If $p(x)=f[g(x)]$, then $p'(x)=[f[g(x)]]'=f'(g(x))g'(x)$

Recall: $p'(x)=\lim_{h\to 0}\frac{p(x+h)-p(x)}{h}$

$p'(x)=[f[g(x)]]'$

$=\lim_{h\to 0}\frac{f[g(x+h)]-f[g(x)]}{h}$

Assuming that $g(x+h)-g(x)\neq 0$, we can write

$p'(x)=[f[g(x)]]'$

$=\lim_{h\to 0}\left[\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{\text{h}}\right]$

$=\lim_{h\to 0}\left[\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\right]\cdot\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$

Since $\lim_{\text{h}\to 0}[g(x+h)-g(x)]=0$, let $g(x+h)-g(x)=k$ and $k\to 0$ as $h\to 0$. We obtain

$=\lim_{k\to 0}\left[\frac{f[g(x)+k]-f[g(x)]}{k}\right]\cdot\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$

Therefore $[f[g(x)]]'=f'(g(x))g'(x)$

$p'(x)=[f[g(x)]]'=f'(g(x))g'(x)$

<u>Chain Rule:</u>

If y is a function of u and u is a function of x (so that y is a composite function), then $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ are both exist.