THINKING

8.The number of bacteria in a certain culture varies as a function of time according to the formula $N(t) = N_0 e^{-0.4t}$ where N_0 represents the quantity present at time t in hours. Show that the rate of change of N with respect to t is proportional to N (i.e. that they are related by some constant factor) [3]

$N(t) = N_0 e^{-0.4t}$

$N'(t) = -0.4 N_0 e^{-0.4t}$

$\underbrace{N_0 e^{-0.4t}}_{= N(t)}$

$N'(t) = -0.4\, N(t)$

↑ constant

The rate of change of N with respect to t is $-0.4 N_0 e^{-0.4t}$ since N_0 is a constant (the original amount)

the only difference between $N'(t)$ and $N(t)$ is a factor of -0.4

$\therefore$ $N'(t) = -0.4\, N(t)$ and they are proportional and related by a constant of -0.4

9. If $f(x) = x^p(1-x)^q$ (where p and q are integers, $p \geq 2, q \geq 2, p \neq q$) show that the critical numbers are $x = 0, x = 1, x = \dfrac{p}{p+q}$. [4]

$f(x) = x^p(1-x)^q$

$f'(x) = px^{p-1}(1-x)^q + x^p(q)(1-x)^{q-1}(-1)$

$= px^{p-1}(1-x)^q - x^p q(1-x)^{q-1}$

$= x^p(1-x)^q\left(\frac{p}{x} - \frac{q}{1-x}\right)$

set $f'(x) = 0$ to find critical points

$x^p = 0$ or $(1-x)^q = 0$ or $\frac{p}{x} - \frac{q}{1-x} = 0$

$x = 0$ or $1 - x = 0$

$x = 1$

$p(1-x) - xq = 0$

$p - px - xq = 0$

$p - x(p+q) = 0$

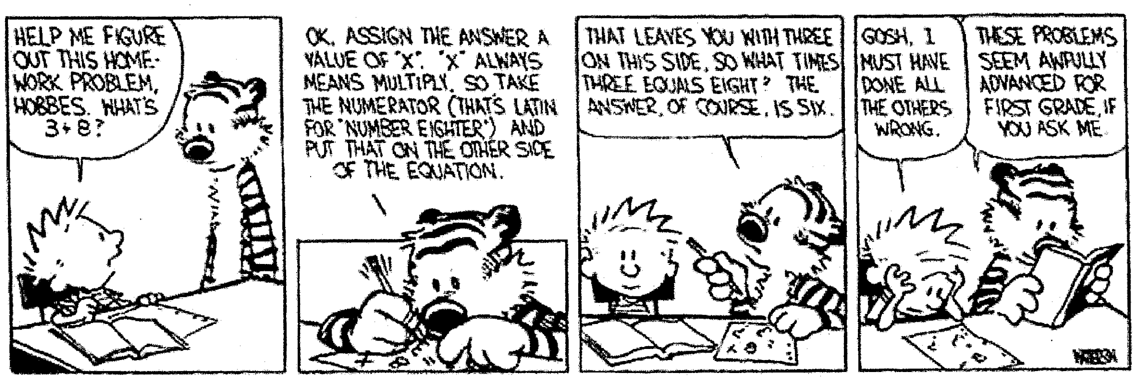
$-x(p+q) = -p$

$x(p+q) = p$

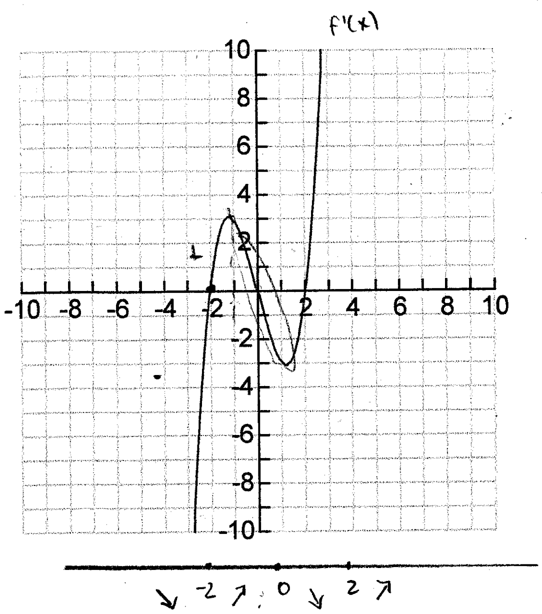
$x = \dfrac{p}{p+q}$

$\therefore$ The critical numbers are $x = 0$, $x = 1$, $x = \dfrac{p}{p+q}$

7



3. The function f'(x) (the <u>DERIVATIVE</u> function) is shown below. State values of x for the following. (Use approximate values of x where necessary.) [6]



a) $f''(x) > 0$ $x < -1.2$, $x > 1.2$

b) $f(x)$ has a local minimum

$x = -2$, $x = 2$

c) $f(x)$ has point(s) of inflection

$x = 1.2$ $x = -1.2$

d) the interval(s) over which $f(x)$ is increasing

$-2 < x < 0$, $x > 2$

4. For each function determine the equations of any vertical, horizontal or oblique asymptotes. Evaluate f(x) as $x \to \pm\infty$. Evaluate the one sided limits around the vertical asymptotes. [7]

a) $y = \dfrac{x}{x+1}$

/3

V.A set denominator equal to 0

$x+1=0$

$x=-1$

No oblique

$\lim_{x \to -1^-} f(x) = +\infty$

$\lim_{x \to -1^+} f(x) = -\infty$

H.A

$\lim_{x \to \infty} \frac{x}{x+1}$

$= \lim_{x \to \infty} \dfrac{\frac{x}{x}}{\frac{x}{x} + \frac{1}{x}}$

$= \frac{1}{1+0}$

$= 1$

∴ H.A at $y = 1$

Test

$f(100) = 0.99$

$f(-100) = 1.01$

∴ as $x \to +\infty$ $f(x) \to 1^-$

as $x \to -\infty$ $f(x) \to 1^+$

b) $y = \dfrac{3x^2 - 8x - 7}{x-4}$

/4

V.A set denominator equal to 0

$x - 4 = 0$

$x = 4$

$\lim_{x \to 4^+} f(x) = +\infty$

$\lim_{x \to 4^-} f(x) = -\infty$

No H.A

$$\begin{array}{r} 3x + 4 \\ x-4 \overline{)\,3x^2 - 8x - 7} \\ \underline{3x^2 - 12x} \\ 4x - 7 \\ \underline{4x - 16} \\ 9 \end{array}$$

∴ $f(x) = 3x + 4 + \dfrac{9}{x-4}$

Test

Let $g(x) = \frac{9}{x-4}$

$g(1000) = 9.03 \times 10^{-3}$

$g(-1000) = -8.96 \times 10^{-3}$

∴ as $x \to \infty$ $f(x) \to$ $3x+4$ from above since $\frac{9}{x-4} \to 0^+$

as $x \to -\infty$ $f(x) \to 3x+4$ from below since $\frac{9}{x-4} \to 0^-$

C ∴ $y = 3x+4$ is oblique asymptote

25K

5. Determine the equation of the tangent to the curve $y = \cos 2x$ at $x = \frac{\pi}{6}$. Express your answer in exact form. [5]

$y = \cos 2x$
$= 1 - 2\sin^2 x$
$m_{tan} = \frac{dy}{dx}$
$= -4\sin x \cos x$

y at $x = \frac{\pi}{6} = \cos 2\left(\frac{\pi}{6}\right)$
$= \frac{1}{2}$ ∴ point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ is on tagent

m_{tan} at $x = \frac{\pi}{6}$
$= -4\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$
$= -4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
$= -\sqrt{3}$

∴ equation is of the form
$y = -\sqrt{3}(x) + b$
sub in $\left(\frac{\pi}{6}, \frac{1}{2}\right)$
$\frac{1}{2} = -\sqrt{3}\left(\frac{\pi}{6}\right) + b$
$b = \frac{1}{2} + \frac{\pi\sqrt{3}}{6} = \frac{3 + \pi\sqrt{3}}{6}$

∴ ~~equation of the line~~ is
$y = -\sqrt{3}(x) + \frac{3 + \pi\sqrt{3}}{6}$

6. The number of bacteria in a petri dish is doubling every minute, given by $N(t) = 150(2)^t$. At what time, to the nearest tenth of a minute, is the bacteria population increasing at a rate of 48 000/min? [4]

rate = 1st derivative
$N'(t) = 150(2^t)\ln 2$
set $N'(t) = 48000$
$48000 = 150(2^t)(\ln 2)$
$(\ln 2)(2^t) = 320$
$2^t = \frac{320}{\ln 2}$
$t\ln 2 = \ln\left(\frac{320}{\ln 2}\right)$
$t = \frac{\ln 320 - \ln(\ln 2)}{\ln 2}$
$t \doteq 8.85$
$t = 8.9$

∴ the bacteria is increasing at a rate of 48000/min at 8.9 minutes.

4

9A

7. Sketch the graph of the function $y = x - 2\cos x$, $0 \le x \le 2\pi$ after determining the following:

[14 marks]

a) the endpoints of the interval.
b) local maxima/minima.
c) the intervals of increase/decrease.
d) point(s) of inflection.
e) the intervals of concavity.
f) the equations of any asymptotes.

Round the coordinates of local extrema and inflection points to one decimal place.

LABEL ALL SPECIAL FEATURES ON YOUR GRAPH.

a) $f(0) = 0 - 2\cos 0 = -2 \quad \therefore (0, -2)$ ✓

$f(2\pi) = 2\pi - 2\cos(2\pi) = 2\pi - 2 \approx 4.3 \therefore (2\pi, 4.3)$

b) $f'(x) = 1 - 2(-\sin x)$

$= 1 + 2\sin x$

$0 = 1 + 2\sin x$

$\sin x = -\frac{1}{2}$

$x = \pi + \frac{\pi}{6}$ or $x = 2\pi - \frac{\pi}{6}$

$= \frac{7\pi}{6}$ $\quad = \frac{11\pi}{6}$

$\therefore$ max @ $x = \frac{7\pi}{6}$

min @ $x = \frac{11\pi}{6}$

c)

x	$\frac{3\pi}{6}$	$\frac{7\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$
$f'(x)$	+	0	−	0	+

$\therefore$ increasing @ $0 < x < \frac{7\pi}{6}$, $\frac{11\pi}{6} < x < 2\pi$ ✓

decreasing $\frac{7\pi}{6} < x < \frac{11\pi}{6}$

$f(\frac{7\pi}{6}) = \frac{7\pi}{6} - 2\cos(\frac{7\pi}{6})$ $\quad f(\frac{11\pi}{6}) \approx 4.0$

≈ 5.4 ✓

d) $f''(x) = 0 + 2\cos x = 2\cos x$

$0 = 2\cos x$

$0 = \cos x$

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ ✓

$\therefore$ POI @ $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

$f(\frac{\pi}{2}) = 1.6$

$f(\frac{3\pi}{2}) = 4.7$ ✓

e)

		$\frac{\pi}{2}$		$\frac{3\pi}{2}$	
$f''(x)$	+	0	−	0	+
	∪		∩		∪

✓

$\therefore$ CU @ $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$ ✓

CD @ $\frac{\pi}{2} < x < \frac{3\pi}{2}$

f) VA n/a

OA n/a

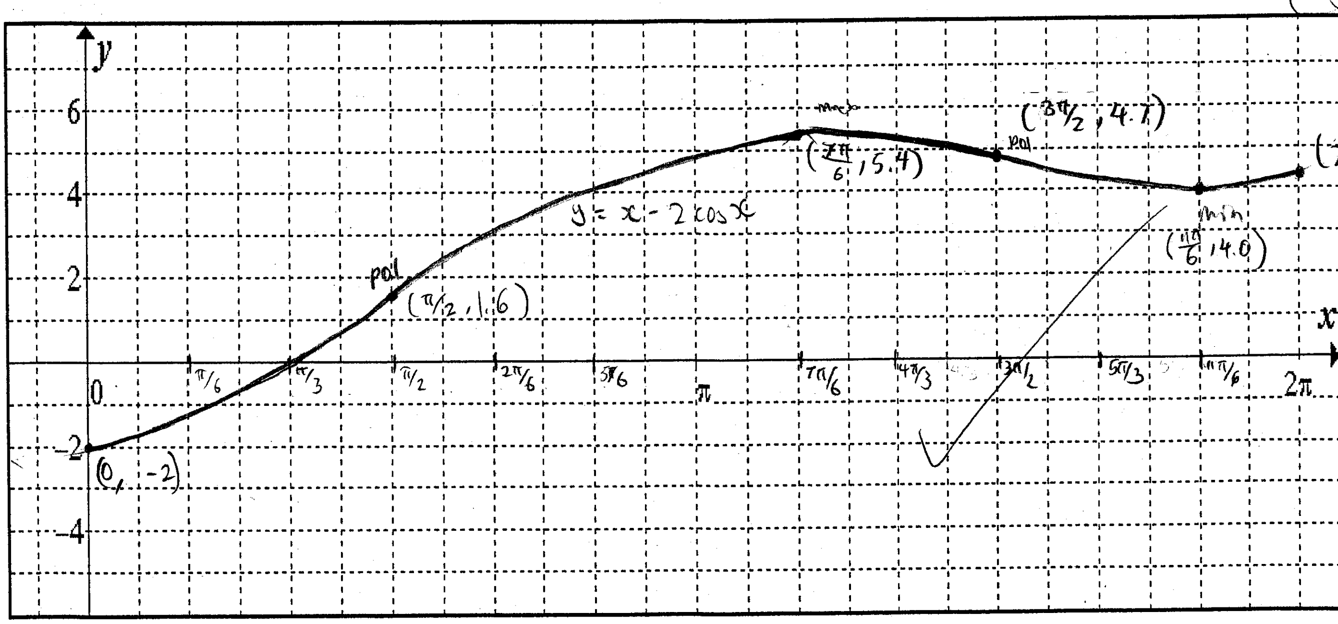
HA $f(x) = \lim_{x \to \infty} x - 2\cos x$ ✓

$= \infty \quad \therefore$ n/a

14 ✓

22A

Great Solution!



7. Sketch the graph of the function $y = x - 2\cos x$, $0 \le x \le 2\pi$ after determining the following: [14 marks]

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Round the coordinates of local extrema and inflection points to one decimal place.

LABEL ALL SPECIAL FEATURES ON YOUR GRAPH.

a) $y(0) = 0 - 2\cos 0 = -2$ ✓
$y(2\pi) = 2\pi - 2\cos(2\pi) \doteq 4.283$ ✓ * rounded on graph

b) $y'(x) = 1 + 2\sin x$
$0 = 1 + 2\sin x$
$-1 = 2\sin x$
$-\frac{1}{2} = \sin x$
$x = \frac{11\pi}{6}, \frac{7\pi}{6}$

		$\frac{7\pi}{6}$		$\frac{11\pi}{6}$	
$y'(x)$	+	0	−	0	+
$y(x)$	↗	local max	↘	local min	↗

✓

min/max $y\left(\frac{7\pi}{6}\right) = \frac{7\pi}{6} - 2\cos\left(\frac{7\pi}{6}\right) \doteq 5.397$ $\left(\frac{7\pi}{6}, 5.397\right)$ ✓ *rounded on graph
$y\left(\frac{11\pi}{6}\right) = \frac{11\pi}{6} - 2\cos\left(\frac{11\pi}{6}\right) \doteq 4.0275$ $\left(\frac{11\pi}{6}, 4.0275\right)$

d) $y''(x) = 2\cos x$
$0 = 2\cos x$
$0 = \cos x$
$x = \frac{\pi}{2}, \frac{3\pi}{2}$

		$\frac{\pi}{2}$		$\frac{3\pi}{2}$	
$y''(x)$	+	0	−	0	+
$y(x)$	cu	poi	cd	poi	cu

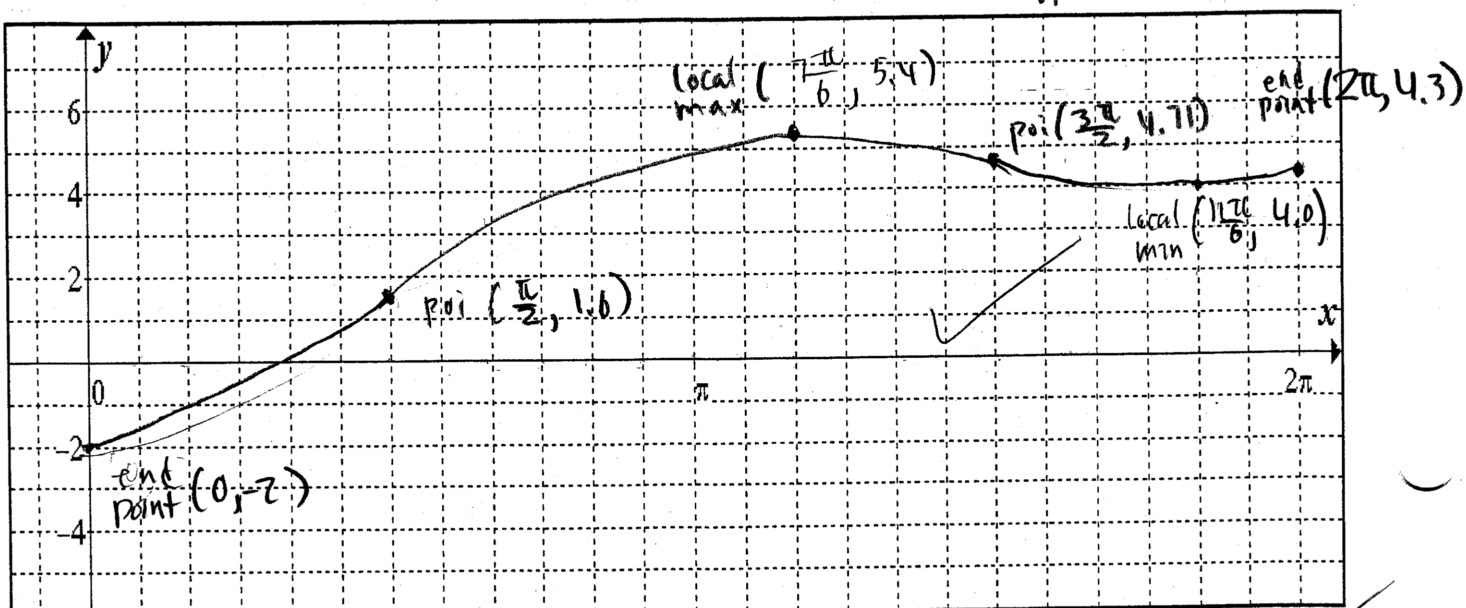
✓

poi $y\left(\frac{\pi}{2}\right) \doteq 1.57$ ✓ * rounded on graph
$y\left(\frac{3\pi}{2}\right) \doteq 4.71$

c) inc: $0 < x < \frac{7\pi}{6}$, $x > \frac{11\pi}{6}$
dec: $\frac{7\pi}{6} < x < \frac{11\pi}{6}$

e) cu: $0 < x < \frac{\pi}{2}$, $\frac{3\pi}{2} < x$ ✓
cd: $\frac{\pi}{2} < x < \frac{3\pi}{2}$

f) *. the function is not reciprocal, thus has no VA, HA and OA. However, the function's characteristic will be similar to $y = x$ as x approaches $\pm\infty$ Good THINKING!



23A

THINKING

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$p(1-x) - xq = 0$

$p - px - xq = 0$

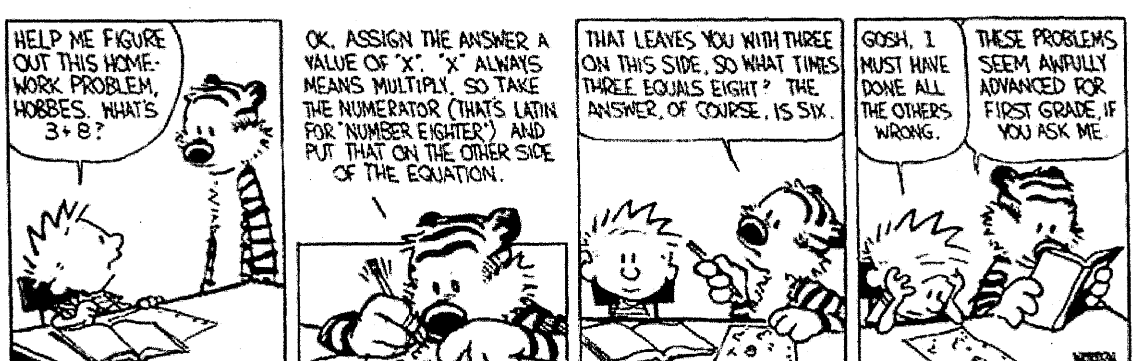
$p - x(p+q) = 0$

$-x(p+q) = -p$

$x(p+q) = p$

$x = \frac{p}{p+q}$

$\therefore$ The critical numbers are $x = 0$, $x = 1$, $x = \frac{p}{p+q}$



7

N		Date:
Answer all questions in pencil. Show all steps for full marks. Express final answers in exact form unless instructed otherwise. Communication will be evaluated on the use of proper mathematical form and evidence of work shown.	Mark: K: 25/25 C: 5/5 A: 23/23 TIPS: 7/7	

KNOWLEDGE

1. Are the following statements *always* true or false? [4]

a) If $f'(2)=0$ then $f(x)$ has a local maximum or minimum when $x=2$. F ✓

b) If $f(x)$ is a polynomial function of degree 3 then its graph has one point of inflection. T ✓

c) If $f''(1)=0$ then the graph of $f(x)$ has a point of inflection when $x=1$. F ✓

d) If $f(x)$ is a rational function then is graph has a horizontal asymptote. F ✓

2. Find $\frac{dy}{dx}$ for each of the following functions. Express your answer in fully factored form. [8]

a) $y=\tan x\cos x$

$$\frac{dy}{dx}=\sec^2 x\cos x-\sin x\tan x$$
$$=\frac{1}{\cos x}-\frac{\sin^2 x}{\cos x}$$
$$=\frac{\cos^2 x}{\cos x}$$
$$=\cos x$$ ✓

b) $y=\frac{x}{e^x}$

$$\frac{dy}{dx}=\frac{e^x-xe^x}{e^{2x}}$$
$$=\frac{e^x(1-x)}{e^{2x}}$$
$$=\frac{1-x}{e^x}$$ ✓

c) $y=\sqrt{\sin^2 x-\cos^2 x}$

$$=(\sin^2x-\cos^2x)^{\frac{1}{2}}$$
$$\frac{dy}{dx}=\frac{1}{2}(\sin^2x-\cos^2x)^{-\frac{1}{2}}(2\sin x\cos x+2\cos x\sin x)$$
$$=\frac{4\sin x\cos x}{2\sqrt{\sin^2x-\cos^2x}}$$
$$=\frac{2\sin x\cos x}{\sqrt{\sin^2x-\cos^2x}}$$ ✓

d) $y=e^{\sin^2 x}$

$$\frac{dy}{dx}=2\sin x\cos x\cdot e^{\sin^2 x}$$ ✓