Name: ____________________ CHAPTER 6&7 TEST *VECTORS* KEY

Date: ____________________

K: /19 A: /23 C: /10 T: /8

Show solutions of good mathematical form using the methods of this course for full marks.
All angles should be stated in degrees.
Final answers should be in exact form unless otherwise stated.

Part A: KNOWLEDGE AND UNDERSTANDING

1. Given $\vec{a} = [1, 2, -1]$, $\vec{b} = [3, 6, 2]$ and $\vec{c} = [2, -2, 1]$, find:

a) $\vec{a} \cdot (\vec{a} + \vec{b})$

[2, 2]

$= (1,2,-1)\cdot[(1,2,-1)+(3,6,2)]$
$= (1,2,-1)\cdot(4,8,1)$
$= 4+16-1$
$= 19$ ✓✓

b) unit vector of $\vec{c}$

$|\vec{c}| = \sqrt{2^2+(-2)^2+1}$
$= \sqrt{9}$
$= 3$
$\therefore \hat{c} = \frac{1}{3}(2,-2,1)$ or $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ ✓✓

c) *the vector* $proj_{\vec{c}}\vec{b}$

[2, 2]

$= \dfrac{\vec{b}\cdot\vec{c}}{|\vec{c}|}\hat{c}$

$= \dfrac{(3,6,2)\cdot(2,-2,1)}{3}\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

$= \dfrac{6-12+2}{3}\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

$= -\frac{4}{3}\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) = \left(-\frac{8}{9}, \frac{8}{9}, -\frac{4}{9}\right)$ OR $-\frac{4}{9}(2,-2,1)$ ✓✓

d) $\vec{a}\times\vec{c}$ $= (1,2,-1)\times(2,-2,1)$

1 2 −1 1
2 −2 1 2

$= (2-2, -2-1, -2-4)$
$= (0, -3, -6)$ ✓✓

2. Given parallelogram GHIK with vertices G(3, –1, 1), H(5, 0, –2), and K(6, 3, –4).
(The vertices are taken in the given order)

a) Determine the coordinates of vertex I.

[3, 3]

Let $I = (x, y, z)$
$\vec{GH} = \vec{KI}$
$(5,0,-2)-(3,-1,1) = (x,y,z)-(6,3,-4)$
$(2,1,-3) = (x-6, y-3, z+4)$
$\therefore x-6=2$ $y-3=1$ $z+4=-3$
$\boxed{x=8}$ $\boxed{y=4}$ $\boxed{z=-7}$
$\therefore I = (8,4,-7)$ ✓✓✓

b) Determine $\angle K$ **(one decimal place).**

$\vec{KG} = (-3,-4,+5)$
$\vec{KI} = (2,1,-3)$
$\vec{KG}\cdot\vec{KI} = |\vec{KG}||\vec{KI}|\cos\theta$
$\therefore \cos\theta = \dfrac{(-3,-4,+5)\cdot(2,1,-3)}{\sqrt{9+16+25}\sqrt{4+1+9}}$
$= \dfrac{-6-4-15}{\sqrt{50}\sqrt{14}}$
$= \dfrac{-25}{5\cdot 2\sqrt{7}}$
$\cos\theta = \dfrac{-5}{2\sqrt{7}}$
$\theta \doteq 160.9°$ ✓✓✓

3. If $|\vec{u}| = 15$, $|\vec{v}| = 11$, and the angle between $\vec{u}$ and $\vec{v}$ is 25°, find $|proj_{\vec{v}}\vec{u}|$. **(one decimal place)**

[2]

$|proj_{\vec{v}}\vec{u}| = |\vec{u}|\cos\theta$
$= 15\cos 25°$
$\doteq 13.6$ ✓✓

4. Given the points P(1, 2, –1) and Q(0, 3, –2), find a unit vector collinear with $\overrightarrow{PQ}$.

[3]

$\overrightarrow{PQ} = (0, 3, -2) - (1, 2, -1)$

$\overrightarrow{PQ} = (-1, 1, -1)$ ✓

$|\overrightarrow{PQ}| = \sqrt{1+1+1}$

$= \sqrt{3}$ ✓

$\therefore \hat{PQ} = \frac{1}{\sqrt{3}}(-1, 1, -1)$

or ✓ $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

Part B: <u>APPLICATION</u>

5. Determine the area of $\triangle ABC$ with vertices A(2, –1, 4), B(3, 1, –1), and C(1, 0, 2).

[3]

$\overrightarrow{AB} = (1, 2, -5)$ ✓

$\overrightarrow{AC} = (-1, 1, -2)$

$\overrightarrow{AB} \times \overrightarrow{AC} = (-4+5, 5+2, 1+2)$

$= (1, 7, 3)$ ✓

$Area_{\triangle ABC} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{1}{2}\sqrt{1+49+9}$ ✓

$= \frac{\sqrt{59}}{2}$ units squared.

$\overrightarrow{BC} = (-2, -1, 3)$

6. The angle between vectors $\vec{a} = [3,1,0]$ and $\vec{b} = [1,1,p]$ is 45°. Find possible value(s) for p.

[4]

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ ✓

$(3,1,0)\cdot(1,1,p) = \sqrt{10}\sqrt{1+1+p^2}\cos 45°$

$3+1 = \sqrt{10}\sqrt{p^2+2}\,\frac{\sqrt{2}}{2}$ ✓

$4 = \frac{2\sqrt{5}}{2}\sqrt{p^2+2}$

$16 = 5(p^2+2)$

$5p^2 + 10 = 16$

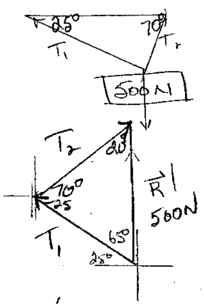
$p^2 = \frac{6}{5}$ ✓

$p = \pm\sqrt{\frac{6}{5}}$ ✓

7. A 500N sign is suspended by two cables that make angles of 25° and 70° with the horizontal. Determine the tension in each cable. **Provide a labeled vector diagram. Include "Let… " statements.**

[4]

Let T_1 and T_2 represent the tension in each cable respectively. ✓



✓

$\frac{T_1}{\sin 20°} = \frac{T_2}{\sin 65°} = \frac{500}{\sin 95°}$

$T_1 = \frac{500\sin 20°}{\sin 95°}$

$\doteq 171.7N$ ✓

$T_2 = \frac{500\sin 65°}{\sin 95°}$

$\doteq 454.9N$ ✓

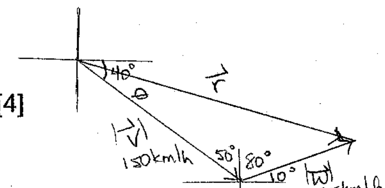
8. A light airplane is flying at 150 km/h [E40°S]. The wind is blowing at 25 km/h from [W10°S]. Find the resultant groundspeed and direction of the plane. **Provide a labeled vector diagram. Include "Let… " statements. Express your answers correct to one decimal place.**

Let $\vec{v}$ represent the plane's velocity

" $\vec{w}$ " " wind velocity

" $\vec{r}$ " " resultant velocity.

[4]



✓

$|\vec{r}| = \sqrt{150^2 + 25^2 - 2(25)(150)\cos 130°}$

$\doteq 167.2$ km/h ✓

$\frac{\sin\theta}{25} = \frac{\sin 130°}{167.2}$

$\theta = \sin^{-1}\left[\frac{25\sin 130°}{167.2}\right] \doteq 6.6°$ ✓

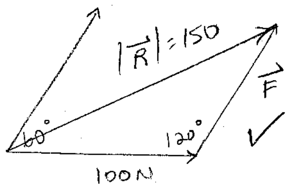
DIRECTION:

$40° - 6.6° = 33.4°$

$\therefore$ The resultant groundspeed is 167.2km/h [E 33.4° S] ✓

9. Two forces act on an object at an angle of 60° to each other. One force has a magnitude of 100N and the resultant has a magnitude of 150N. Determine the magnitude of the second force. **Provide a labeled vector diagram. Include a "Let… " statement. Express your answer correct to one decimal place.**

[4]



Let $\vec{F}$ represent the required force

$150^2 = 100^2 + |\vec{F}|^2 - 2(100)(|\vec{F}|)\cos 120°$ ✓

$22500 = 10000 + |\vec{F}|^2 - 200|\vec{F}|(-\frac{1}{2})$

$|\vec{F}|^2 + 100|\vec{F}| - 12500 = 0$

$|\vec{F}| = \dfrac{-100 \pm \sqrt{100^2 - 4(-12500)}}{2}$ ✓

$\doteq 72.5$ or -172.5 reject, $|\vec{F}| > 0$ ✓

∴ The magnitude of the second force is about 72.5 Newtons.

OR use SINE LAW TWICE

$|\vec{F}| \doteq 72.4$ N with rounding

10. Find the direction angles $\alpha, \beta, and\ \gamma$ that vector $\vec{PQ} \doteq (2,1,4)$ makes with each of the co-ordinate axes.

[4]

$|\vec{PQ}| = \sqrt{4+1+16}$

$|\vec{PQ}| = \sqrt{21}$ ✓

$\therefore \hat{PQ} = \left(\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right)$

$\therefore \cos\alpha = \frac{2}{\sqrt{21}}$ $\qquad \cos\beta = \frac{1}{\sqrt{21}}$ $\qquad \cos\gamma = \frac{4}{\sqrt{21}}$

$\alpha \doteq 64.1°$ ✓ $\qquad \beta \doteq 77.4°$ ✓ $\qquad \gamma \doteq 29.2°$ ✓

Part C: COMMUNICATION [7]

i) The addition of two opposite vectors results in the zero vector.

ii) If two vectors are parallel then their dot product equals the product of their magnitudes.

iii) The magnitude of the cross product of two vectors equals the area of the parallelogram formed by the two vectors.

iv) The Commutative property holds for the dot product but not for the cross product.

v) The dot product of any two of the vectors $\vec{i}, \vec{j}, \vec{k}$ is zero since they are (mutually) perpendicular.

vi) The equilibrant vector is the opposite of the resultant vector.

12. Describe the difference between airspeed and groundspeed. [3]

Airspeed is the speed of a plane (helicopter? balloon?) in air that is relative to someone on the plane. ✓
Groundspeed is the speed of a plane relative to someone on the ground ✓ and includes the effect of the wind. ✓

Part D: <u>THINKING AND PROBLEM SOLVING</u> [8 marks]

14. Show that for any two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$: $|\vec{a}\times\vec{b}|=\sqrt{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})^2}$

[4]

$LS = |a \times b|$

$= |\vec{a}||\vec{b}| \sin\theta$ ✓

$RS = \sqrt{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})^2}$

$= \sqrt{|\vec{a}|^2|\vec{b}|^2 - |a|^2|b|^2\cos^2\theta}$

$= \sqrt{|\vec{a}|^2|\vec{b}|^2[1-\cos^2\theta]}$

$= \sqrt{|\vec{a}|^2|\vec{b}|^2\sin^2\theta}$

$= |\vec{a}||\vec{b}|\sin\theta$

✓✓✓

3. Given $\vec{a}=[-5,-2,6]$, $\vec{b}=[2,4,3]$, $\vec{c}=[x,-2,z]$ and $\vec{a}\times\vec{c}=\vec{b}$, find the values of x and z.

[4]

$\vec{a}\times\vec{c} = (-5,-2,6)\times(x,-2,z)$

$-5 \quad -2 \quad 6 \quad -5$

$x \quad -2 \quad z \quad x$

$= (-2z+12, 6x+5z, 10+2x)$

$\because \vec{a}\times\vec{c} = \vec{b}$

$\therefore (-2z+12, 6x+5z, 10+2x) = (2,4,3)$ ✓

① $-2z+12=2$ ② $6x+5z=4$ ③ $10+2x=3$

$\boxed{z=5}$

$2x=-7$

$\boxed{x=-\frac{7}{2}}$ ✓

Check

Sub z & x into ②

LS	RS
$6(-\frac{7}{2})+5(5)$	4
$=-\frac{42}{2}+25$	✓
$=-21+25$	
$=4$	

$LS = RS$

$\therefore \boxed{x=-\frac{7}{2}}$ and $\boxed{z=5}$

✓